RC filter

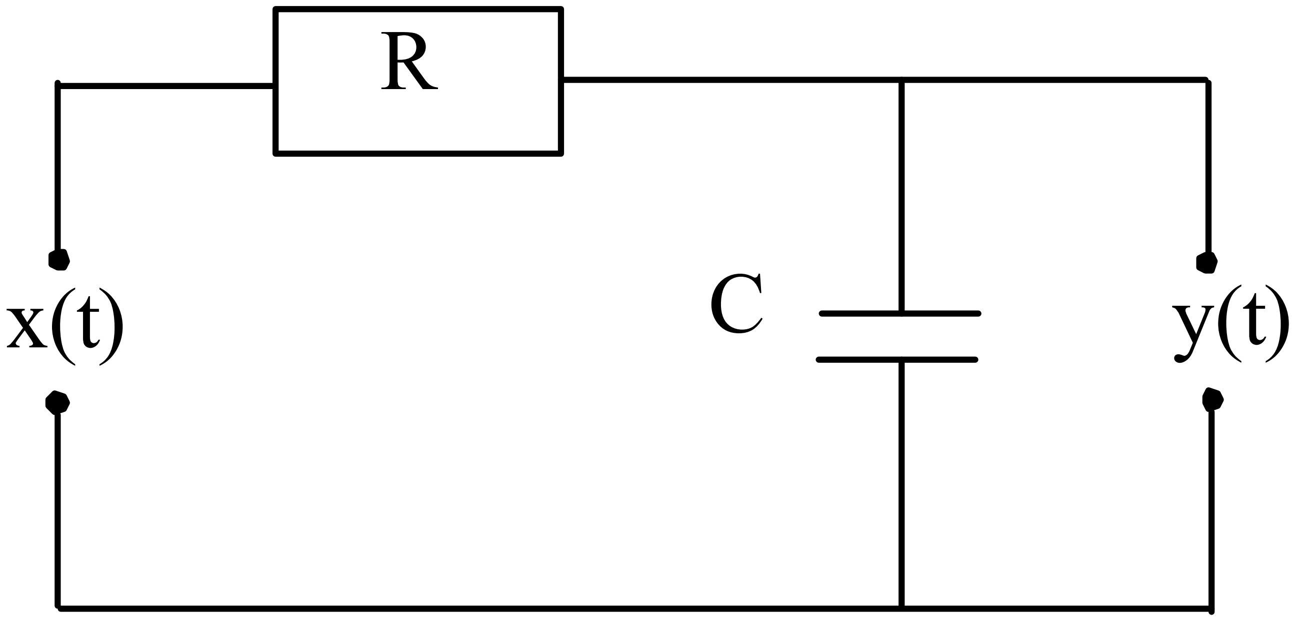
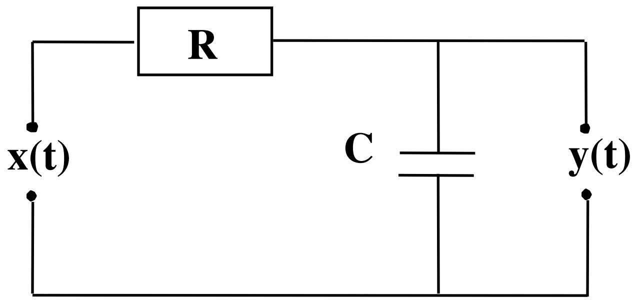


Fig. 2.1 RC filter.

Differential equation



Voltage balance?

$$RI(t) + y(t) = x(t)$$

Current through resistor R and capacitance C:

$$I(t) = C\dot{y}(t)$$

$$\Rightarrow \quad \boxed{RC\dot{y}(t) + y(t) - x(t) = 0}$$

First order linear differential equation

Linearity

Input		**Output**
$x_1(t)$	$\longrightarrow$	$y_1(t)$
$x_2(t)$	$\longrightarrow$	$y_2(t)$
$x_3(t) = a\, x_1(t) + b\, x_2(t)$	$\longrightarrow$	$y_3(t) = ?$

Property of linear system: $\Rightarrow$ $y_3(t) = a\, y_1(t) + b\, y_2(t)$

RC filter = linear (time invariant) system = LTI system

Frequency response function and Fourier transform

$$RC\dot{y}(t) + y(t) - x(t) = 0 \qquad \text{for zero input signal x(t)}$$

x(t) = 0: $\Longrightarrow$ $RC\dot{y}(t) + y(t) = 0$

Solution: $y(t) = -\dfrac{1}{RC} \cdot e^{-\frac{t}{RC}}$

What is the solution to arbitrary input signals ?

Approach: Solve for harmonic signals,
then construct arbitrary signals from harmonic signals.

$\Rightarrow$ „Fourier transform“

Fourier transform

Definition: $$\mathcal{F}\left\{x(t)\right\} = X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}\,dt$$

Common practice to write (with $\omega = 2\pi f$): $$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$$

Back transformation: $$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\,d\omega$$

Transform properties of the Fourier transform

($\Leftrightarrow$ indicates a transform pair, x(t) $\Leftrightarrow X(j\omega)$):

- *Time shifting —* $x(t-a) \Leftrightarrow X(j\omega)\cdot e^{-j\omega a}$
- *Derivative —* $\dfrac{d}{dt}x(t) \Leftrightarrow j\omega \cdot X(j\omega)$
- *Integration —* $\displaystyle\int_{-\infty}^{\infty} x(t)dt \Leftrightarrow \frac{1}{j\omega}\cdot X(j\omega)$

Convolution

$$g(t) = f(t) * h(t) = \int_{-\infty}^{\infty} f(\tau) \cdot h(t-\tau)\,d\tau$$

• *Convolution Theorem —*

$$\mathrm{f(t)} * \mathrm{h(t)} \Leftrightarrow \mathrm{F(j\omega)} \cdot \mathrm{H(j\omega)}$$

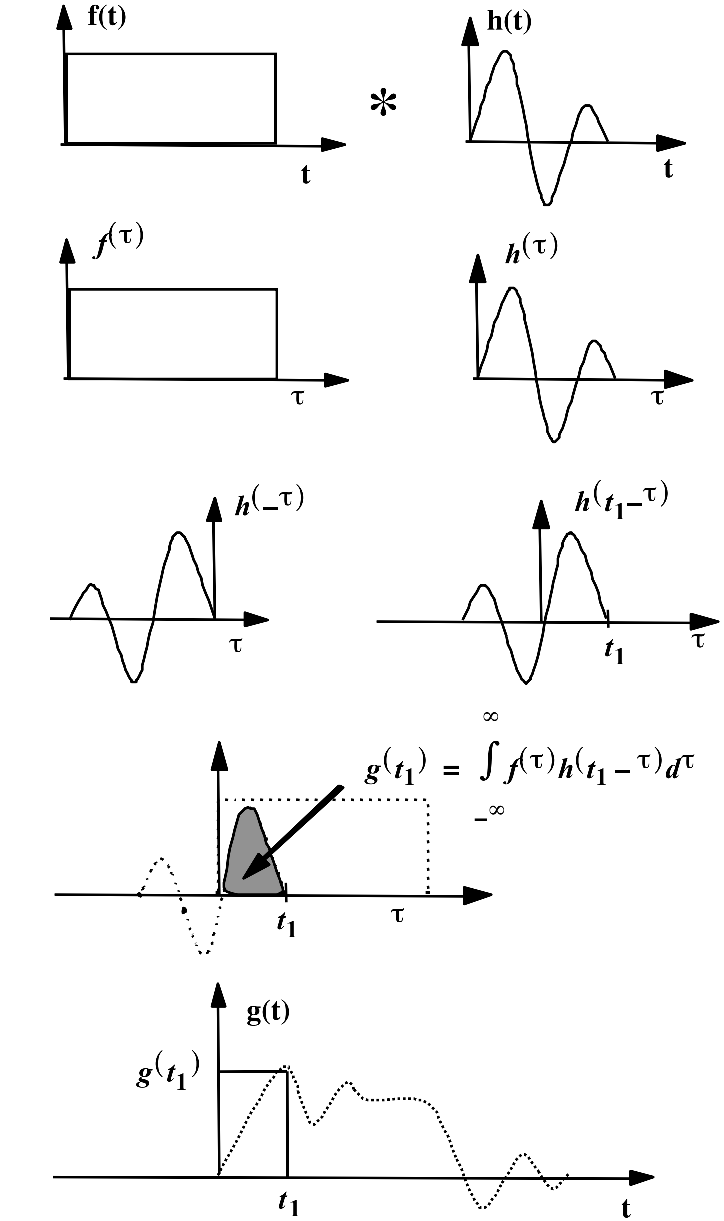


Fig. 2.5 Graphical interpretation of the convolution operation.

Harmonic input signals

$$RC\dot{y}(t) + y(t) - x(t) = 0$$ for harmonic input signal x(t)

Harmonic input signal: $x(t) = A_i e^{j\omega t}$

Ansatz for the output signal: $y(t) = A_o e^{j\omega t}$

$$\dot{y}(t) = j\omega A_o e^{j\omega t}$$

$$\Longrightarrow \quad A_o e^{j\omega t}(RCj\omega + 1) = A_i e^{j\omega t}$$

$$\Longrightarrow \quad \boxed{\frac{A_o}{A_i} = \frac{1}{RCj\omega + 1} = T(j\omega)}$$

Frequency response function

Input/output relation

$$\boxed{A_o = T(j\omega) \cdot A_i}$$

Polar form

$$T(j\omega) = \frac{1}{RCj\omega + 1} \longrightarrow \text{polar form}$$

Identities: $\dfrac{1}{\alpha + j\beta} = \dfrac{\alpha}{\alpha^2 + \beta^2} - j\dfrac{\beta}{\alpha^2 + \beta^2}$ and $\Phi = \arctan\left(\dfrac{\mathrm{Im}}{\mathrm{Re}}\right)$

For: $\dfrac{1}{\alpha + j\beta}$: $\Phi = \arctan\left(\dfrac{-\beta}{\alpha}\right)$ and $|...| = \dfrac{1}{\sqrt{1 + (RC\omega)^2}}$

and $T(j\omega) = \dfrac{1}{\sqrt{1 + (RC\omega)^2}} \cdot e^{j\Phi(\omega)}$

$$\Phi(\omega) = \arctan(-RC\omega) = -\arctan(RC\omega)$$

Problem 2.1

Proof that the output signal of the RC filter for a sinusoidal input signal $A_i \sin\,(\omega_0 t)$ is again a sinusoidal signal and determine its frequency and phase. Make use of Euler's formula ($\sin y = (e^{jy} - e^{-jy})/2j$) and the two equations:

$$T(j\omega) = \frac{1}{\sqrt{1 + (RC\omega)^2}} \cdot e^{j\Phi(\omega)}$$

$$\Phi(\omega) = \arctan(-RC\omega) = -\arctan(RC\omega)$$

Use $|T(j\,(-\,\omega))| = |T(j\,(\omega))|$ and $\Phi(-\,\omega) = -\Phi(\omega)$

The frequency response function and the eigenvalue / eigenvector concept

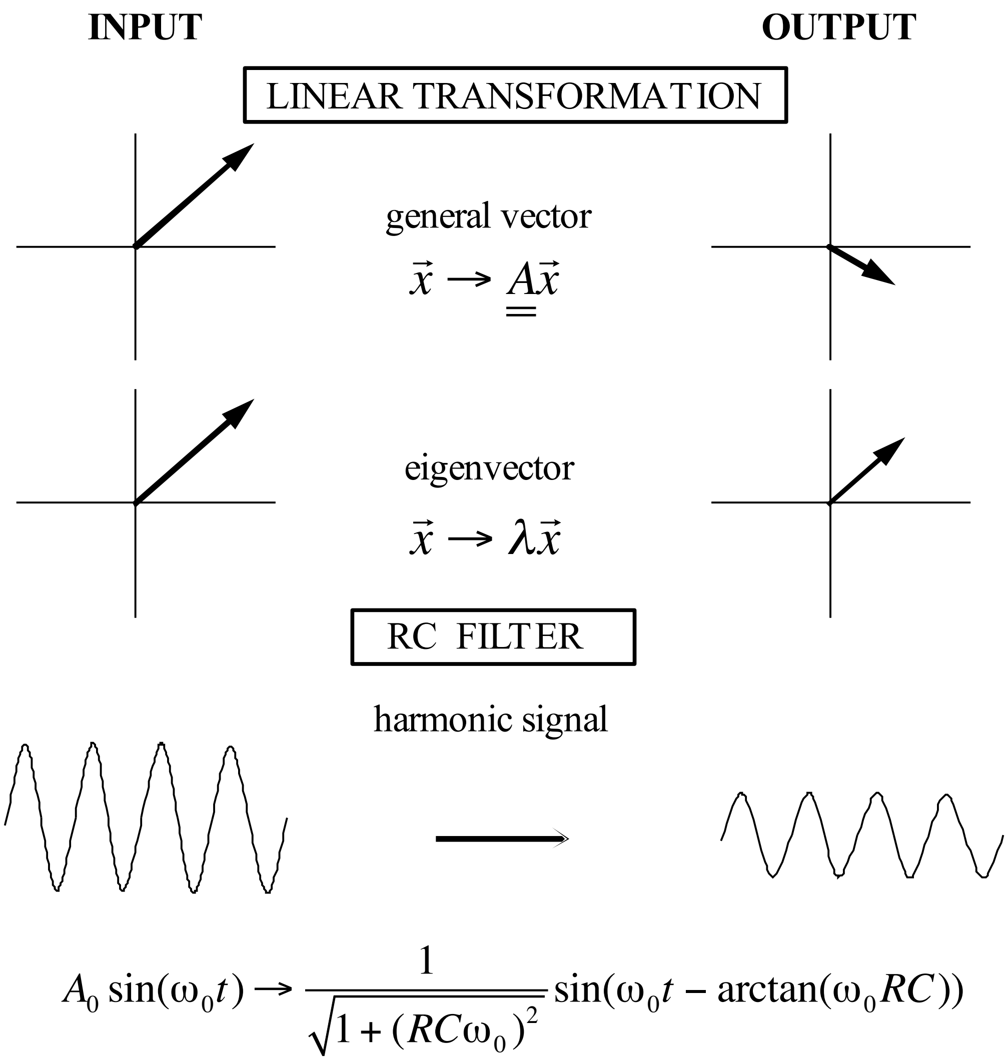


Fig. 2.6 Frequency response function and the eigenvector/eigenvalue concept.

The frequency response and arbitrary input signals

$A_i(j\omega) \rightarrow$ harmonic component of the Fourier spectrum $X(j\omega)$ (input)

$A_o(j\omega) \rightarrow$ harmonic component of the Fourier spectrum $Y(j\omega)$ (output)

The frequency response function relates the Fourier spectrum of the output signal $Y(j\omega)$ to the Fourier spectrum of the input signal X(jω) :

$$T(j\omega) = \frac{Y(j\omega)}{X(j\omega)}$$

• ***Definition* —** The frequency response function $T(j\omega)$ is defined as the Fourier transform of the output signal divided by the Fourier transform of the input signal.

Input/output relation

Fourier spectrum of the filter output:

$$Y(j\omega) = T(j\omega) \cdot X(j\omega)$$

The frequency response function can be measured by comparing output and input signals to the system **without further knowledge of the physics going on inside the filter**!

Transfer function and Laplace transform

Bilateral Laplace transform of *f(t):* $\mathcal{L}\left[f(t)\right] = \int_{-\infty}^{\infty} f(t)e^{-st}dt$

with the complex variable $s = \sigma + j\omega$

$\mathcal{L}\left[f(t)\right]$ will be written as $F(s)$.

Property: $\mathcal{L}\left[\dot{f}(t)\right] = s \cdot F(s)$

Transfer function and Laplace transform

Transforming equation $RC\dot{y}(t) + y(t) - x(t) = 0$

using $\mathcal{L}\left[\dot{f}(t)\right] = s \cdot F(s)$ we obtain:

$$RCsY(s) + Y(s) - X(s) = 0$$

with $Y(s)$ and $X(s)$ being the Laplace transforms of $y(t)$ and $x(t)$, respectively.

$$\Longrightarrow \quad T(s) = \frac{Y(s)}{X(s)} = \frac{1}{1 + sRC} = \frac{1}{1 + s\tau}$$ transfer function

Transfer function

• *Definition* — The transfer function *T(s)* is defined as the Laplace transform of the output signal divided by the Laplace transform of the input signal.

Laplace transform of the output signal:

$$Y(s) = T(s)X(s)$$

RC Filter:

$$T(s) = \frac{1}{1 + s\tau}$$

Special cases:

a) $s \rightarrow j\omega \;\Rightarrow\; T(s) \rightarrow T(j\omega)$

b) $s \rightarrow -\frac{1}{\tau} \;\Rightarrow\; T(s) \rightarrow \infty$ **"pole"**

The impulse response function

Dirac delta 'function' $\delta(t)$ $\quad \int_{-\infty}^{\infty} f(t)\delta(t-\tau)dt = f(\tau)$

Generation:

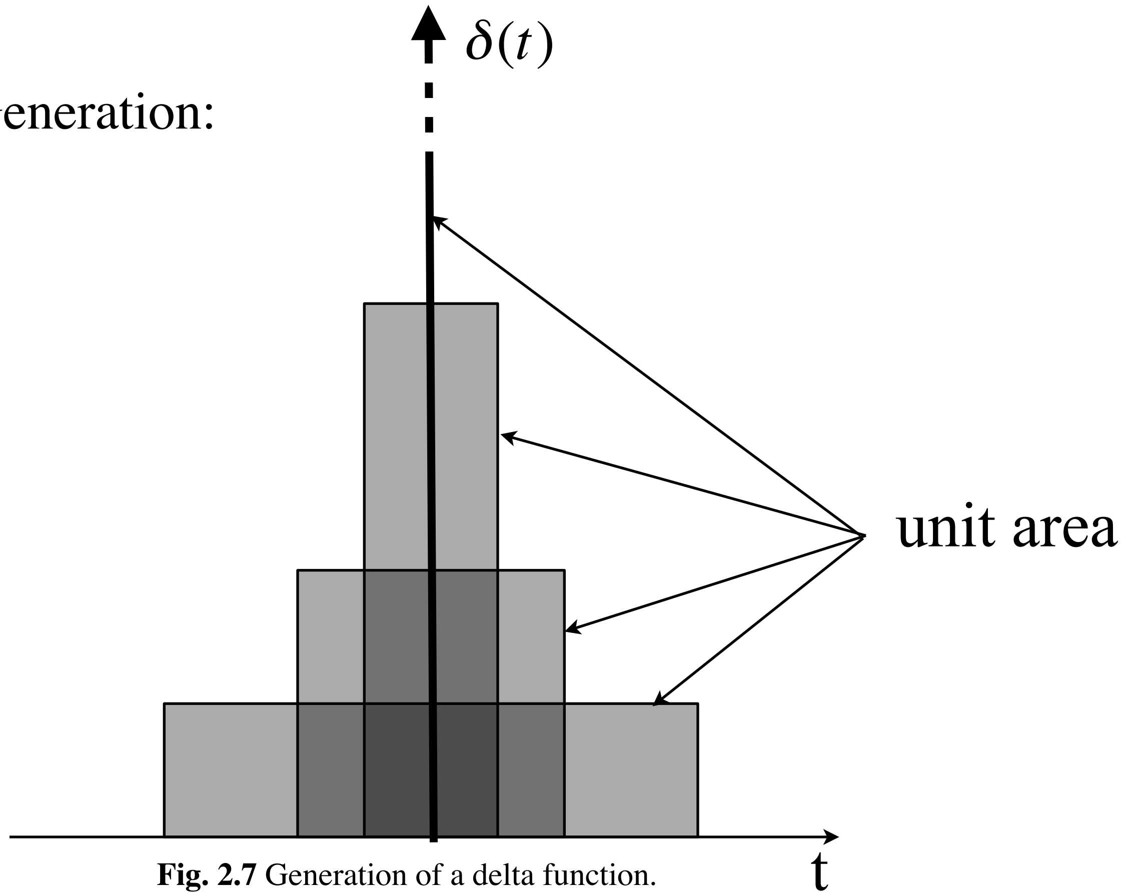


Fig. 2.7 Generation of a delta function.

From the conditions of unit area and infinitesimal duration we obtain

$$\int_{-\infty}^{\infty} \delta(t)dt = 1$$

and $\quad \delta(t) = 0 \;$ for $\; t \neq 0$

Properties $\quad \mathsf{F}\left\{\delta(t)\right\} = \int_{-\infty}^{\infty} \delta(t) \cdot e^{-j2\pi ft} dt = 1$

$$\mathsf{L}\left\{\delta(t)\right\} = \int_{-\infty}^{\infty} \delta(t) \cdot e^{-st} dt = 1$$

The impulse response function

Response of a filter to an impulsive (delta function) input signal.

Properties:

- The frequency response function $T(j\omega)$ is the Fourier transform of the impulse response function.

- The transfer function $T(s)$ is the Laplace transform of the impulse response function.

Proof:

Consider $T(s)$ for $x(t) = \delta(t)$ and hence $X(s) = 1$. In this case, the output signal $y(t)$ becomes the impulse response function $h(t)$ with $H(s)$ being its Laplace transform.

$$T(s) = \frac{Y(s)}{X(s)} = \frac{Y(s)}{1} = H(s) \qquad \text{for } x(t) = \delta(t)$$

The same argument can be made for the frequency response function

$$T(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{Y(j\omega)}{1} = H(j\omega) \qquad \text{for } x(t) = \delta(t)$$

Fourier spectrum of a filter output signal:

$$Y(j\omega) = T(j\omega) \cdot X(j\omega)$$

Convolution theorem:

$$T(j\omega) \cdot X(j\omega) = H(j\omega) \cdot X(j\omega) \quad \Leftrightarrow \quad h(t) * x(t)$$

Consequences:

Filtering

$$y(t) = h(t) * x(t)$$

$$Y(j\omega) = T(j\omega) \cdot X(j\omega)$$

$$Y(s) = T(s) \cdot X(s)$$

Impulse response function of the RC filter

Transfer function of RC filter: $\dfrac{1}{1+s\tau}$

General type: $\dfrac{K}{s+a}$ with $K = 1/\tau$ and $a = 1/\tau$.

Considering $f(t) = K \cdot e^{-at}u(t)$ with $u(t)$ being the unit step function:

$$F(s) = K\int_{-\infty}^{\infty} e^{-at}e^{-st}u(t)dt = K\int_{0}^{\infty} e^{-(s+a)t}dt = \left[-K\frac{e^{-(s+a)t}}{s+a}\right]_{o}^{\infty}$$

The equation above exists only for Re{s+a} > 0 or Re{s} > Re{-a }where it becomes

$$\frac{K}{s+a}$$

Hence $\dfrac{1}{1+s\tau}$ is the Laplace transform of

$$y(t) = \frac{1}{\tau} e^{-\frac{1}{\tau}t} \quad \text{for } t > 0$$

The region where F(s) the exists is called region of convergence

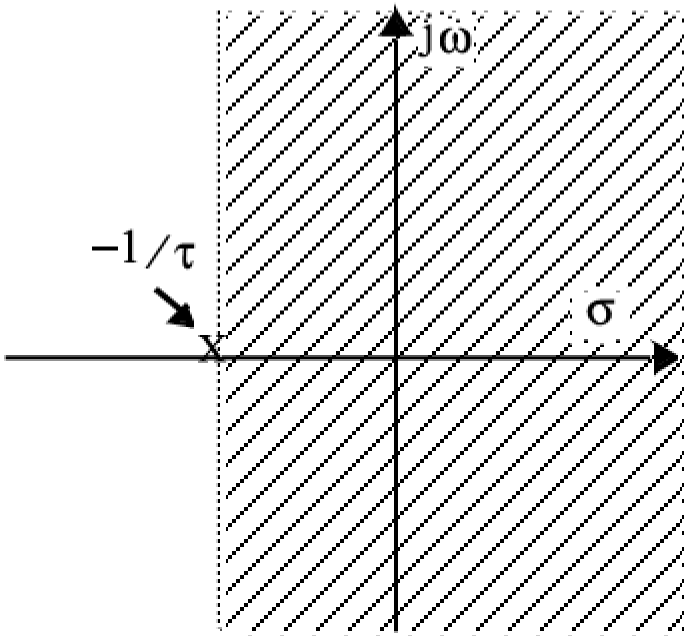


Fig. 2.8 Region of conver gence of F(s)The pole location at $-\dfrac{1}{\tau}$ is marked by an X.

The transfer function of a system is the Laplace transform of its impulse response function.

$y(t) = \frac{1}{\tau} e^{-\frac{1}{\tau}t}$ for $t > 0$ is the impulse response of RC filter.

Alternative solution:

$f(t) = -K \cdot e^{-at} u(-t)$,with $u(-t)$ being the time inverted unit step function:

$$F(s) = -K \int_{-\infty}^{\infty} e^{-at} e^{-st} u(-t) dt = -K \int_{-\infty}^{0} e^{-(s+a)} dt$$

$$= \left[K \frac{e^{-(s+a)t}}{s+a} \right]_{-\infty}^{0}$$

$$= \frac{K}{s+a}$$

Region of convergence: Re{s+a} < 0 or Re{s} < Re{-a}

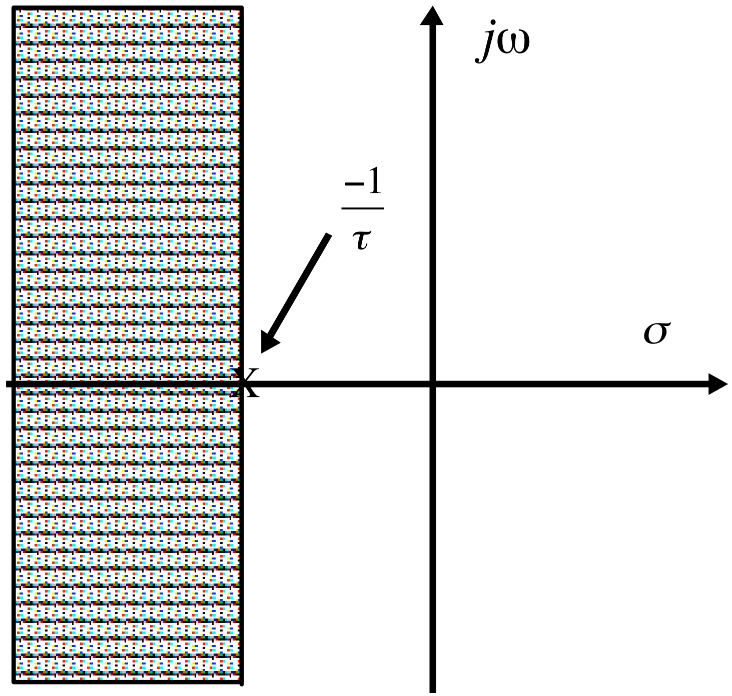


Fig. 1.15 Region of convergence of alternative solution The pole location at $-1/\tau$ is marked by an X.

Impulse response ⇔ Inverse Laplace transform of T(s)

Inverse Laplace transform: $\mathcal{L}^{-1}\left[F(s)\right] = f(t) = \dfrac{1}{2\pi j}\displaystyle\int_{\sigma - j\infty}^{\sigma + j\infty} F(s)e^{st}\,ds$

The path of integration must lie in the region of convergence.

Special case: $s = j\omega$

Inverse Fourier transform

$$f(t) = \frac{1}{2\pi j}\int_{\sigma - j\infty}^{\sigma + j\infty} F(s)e^{st}\,ds = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)e^{j\omega t}\,d\omega$$

ROC and type of impulse response

ROC: right half plane => right-sided impulse response
left half plane => left-sided impulse response

Example: $$F(s) = \frac{K}{s+a}$$

causal: $$f(t) = K \cdot e^{-at}u(t) \;\; \text{for } t > 0$$

anti-causal: $$f(t) = -K \cdot e^{-at}u(-t) \;\; \text{for } t < 0$$

Condition for stability

RC filter: $y(t) = \frac{1}{\tau} e^{-\frac{1}{\tau}t} u(t)$ (physically realizable IR)

time dependence: $-1/\tau \Leftrightarrow$ location of pole of $T(s)$

$$y(t) = \left|s_p\right| \cdot e^{s_p t} u(t)$$ with pole position s_p

case a) pole is located in the left half s plane => causal IR decays

case b) pole is located in right half plane => causal IR grows to infinity

In general:

> In order for a causal system to be stable, all the poles of the transfer function have to be located within the left half of the complex s plane.

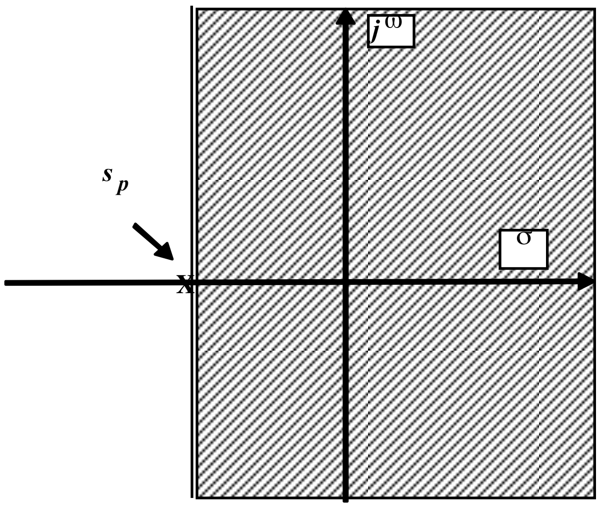
Caution! For anticausal signals the opposite is true. For a pole at $1/\tau$ the anticausal signal $y(-t) = (1/\tau)e^{(1/\tau)(-t)}u(-t)$ would well be stable, although physically unrealizable.

Review

Causal function

$$f_1(t) = K \cdot e^{s_p t} u(t) \quad \text{for } \; t \geq 0$$

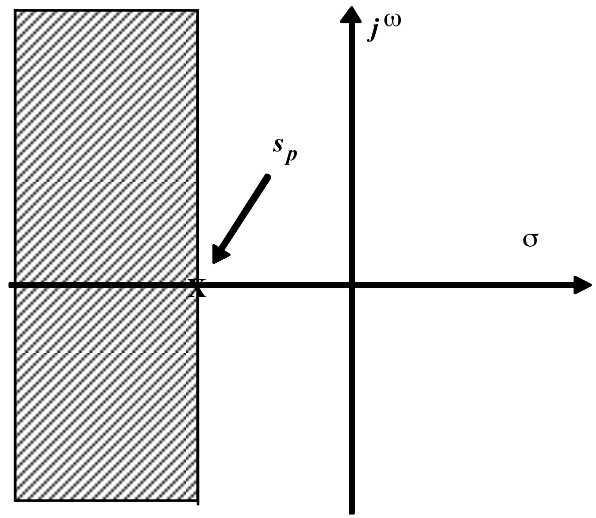
$$\mathcal{L}\left[f_1(t)\right] \text{ exists for } \mathrm{Re}\left\{s\right\} > \mathrm{Re}\left\{s_p\right\}$$



Anti-causal function

$$f_2(t) = -K \cdot e^{s_p t} u(-t) \quad \text{for } \; t < 0$$

$$\mathcal{L}\left[f_2(t)\right] \text{ exists for } \mathrm{Re}\left\{s\right\} < \mathrm{Re}\left\{s_p\right\}$$



$$\boxed{\mathcal{L}\left[f_1(t)\right] = \mathcal{L}\left[f_2(t)\right] = \frac{K}{s - s_p}}$$

Do $f_1(t)$ and $f_2(t)$ have a Fourier transform?

Review

$$\boxed{\mathcal{L}\left[f(t)\right] = \frac{K}{s - s_p}}$$

Impulse response f(t)?

$f(t) = \mathcal{L}^{-1}\left[F(s)\right]$

for any integration path in ROC

or

$f(t) = \mathcal{F}^{-1}\left[F(j\omega)\right]$

ROC has to contain $j\omega$

Depending on intergration path for inverse transform

causal

$f_1(t) = K \cdot e^{s_p t} u(t)$

anti-causal

$f_2(t) = -K \cdot e^{s_p t} u(-t)$

Stability?

$f_1(t) \neq 0$ for t > 0

$f_1(t) = K \cdot e^{s_p t}$ for t > 0

stable for $Re\{s_p\} < 0$

$f_2(t) \neq 0$ for t < 0

$f_2(t) = -K \cdot e^{s_p t}$ for t < 0

stable for $Re\{s_p\} > 0$

The frequency response function and the pole position

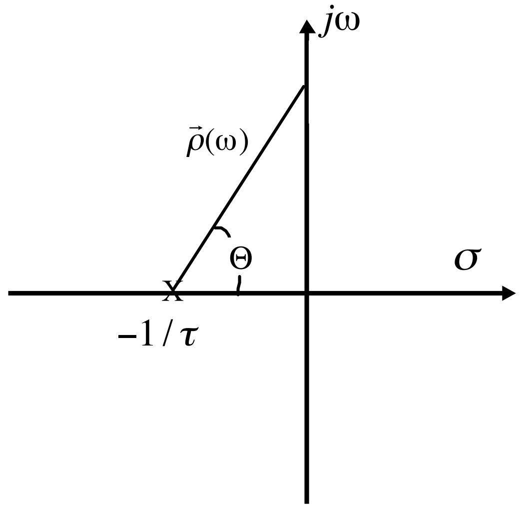


Fig. 2.10 Representation of the RC filter in the s plane. The pole location at -1/τ is marked by an X.

Transfer function RC filter: $$T(s) = \frac{1}{1+s\tau} = \frac{1}{\tau}\left[\frac{1}{(1/\tau)+s}\right]$$

For $s = j\omega$, ω moves along the imaginary axis

$1/\tau + j\omega$ represents the vector $\vec{\rho}(\omega)$

$$\Longrightarrow \quad T(j\omega) = \frac{1}{\tau}\left[\frac{1}{(1/\tau)+j\omega}\right]$$

which is pointing from the pole position towards the actual frequency on the imaginary axis.

in polar coordinates

$$T(j\omega) = \frac{1}{\tau}\left[\frac{1}{\left|\vec{\rho}(\omega)\right| e^{j\Theta}}\right] = \frac{1}{\tau}\left[\frac{1}{\left|\vec{\rho}(\omega)\right|} e^{-j\Theta(j\omega)}\right] = \left|T(j\omega)\right| e^{j\Phi(j\omega)}$$

For the given example, the amplitude value of the frequency response function for frequency ω is proportional to the reciprocal of the length of the vector $\vec{r}(\omega)$ from the pole location to the point $j\omega$ on the imaginary axis. The phase angle equals the negative angle between $\vec{r}(\omega)$ and the real axis.

Problem 2.2

Determine graphically the amplitude characteristics of the frequency response for a RC filter with $R = 4.0$ Ohm and $C = (1.25/2\pi)\ F = 0.1989495\ F$ (1Ohm = 1(V/A), 1F = 1$Asec/V$). Where is the pole position in the S plane? For the plot use frequencies between 0 and 5 Hz.

Problem 2.3

Calculate the frequency response for the RC filter from Problem 2.2 using the Digital Seismology Tutor.

Start up: Digital Seismology Tutor

Shape of frequency response function

$$\left[\text{Corner frequency: } 0,2Hz = 1/5 \cdot \sec^{-1} = 1/(RC \cdot 2\pi)\right]$$

$$|T(j\omega)| = \frac{1}{\tau}\left[\frac{1}{|(1/\tau) + j\omega|}\right] = \frac{1}{\sqrt{1+\omega^2\tau^2}}$$

Definition: $\omega_c := \left|-\frac{1}{\tau}\right| \quad \Rightarrow \quad |T(j\omega)| = \frac{1}{\sqrt{1+\frac{\omega^2}{\omega_c^2}}}$

$$\omega \to 0: \quad |T(j\omega)| \to 1 = const$$

$$\omega \gg \omega_c: \quad |T(j\omega)| \to \omega^{-1}$$

$$Slope_{\log-\log} = \frac{\log_{10} A(\omega_2) - \log_{10} A(\omega_1)}{\log_{10}(\omega_2) - \log_{10}(\omega_1)} = \frac{\log_{10}\left(\frac{A(\omega_2)}{A(\omega_1)}\right)}{\log_{10}\left(\frac{\omega_2}{\omega_1}\right)}$$

$\Rightarrow$ different scale (dB)

Amplitude ratio in *dB* (20 $\log_{10}$(amplitude ratio)):

$$Slope_{dB/D\omega} = 20 \cdot \frac{\log_{10}\left(\frac{A(\omega_2)}{A(\omega_1)}\right)}{\log_{10}\left(\frac{\omega_2}{\omega_1}\right)}$$

for $\left|T(j\omega)\right| >> \omega^{-1}$

amplitude decreases by a factor of 10 over a full decade

Therefore $Slope_{dB/dec} = 20 \cdot \frac{\log_{10}(0,1)}{\log_{10}(10)} = -20dB \quad [dB / decade]$

or following the same argument -6 *dB/octave*

General rule:

> Rule: A single pole in the transfer function causes the slope of the amplitude portion of the frequency response function in a log-log plot to decrease by 20 dB/decade or 6 dB/octave, respectively.

The RC filter and the role of the pole

Transfer function: $T(s) = \dfrac{1}{1 + s\tau}$ pole: $s_p = -\dfrac{1}{\tau}$

- **determines boundary of ROC**
- **position determines stability**
- **length of pole vector determines magnification**

$$\left|T(j\omega)\right| \sim \left|\frac{1}{\rho(\omega)}\right|$$

- **a pole in the transfer function changes the slope of the modulus of the frequency response function by ω^{-1} (20 dB/dec, 6dB/oct) at a corner frequency $\omega_c = \left|s_p\right|$**

Review (to read)

The central theme of this chapter was to study the behaviour of a simple electric RC circuit. We introduced the term filter or system as a device or algorithm which changes some input signal into an output signal. We saw that the RC filter is an example for a **linear, time invariant (LTI) system**, which could be described by a linear differential equation. From the solution of the **differential equation** for harmonic input we obtained the result that the output is again a harmonic signal. We introduced the concept of the **frequency response function** as the Fourier transform of the output signal divided by the Fourier transform of the input signal. The frequency response function was seen to have important properties:

• The values of the frequency response function are the eigenvalues of the system.
• Knowing the frequency response function, we can calculate the output of the filter to arbitrary input signals by multiplying the Fourier transform of the input signal with the frequency response function.
• The frequency response function is the Fourier transform of the impulse response. Knowing the impulse response function, we can calculate the output of the filter to arbitrary input signals by convolving the input signal with the impulse response function.

We then introduced the concept of the **transfer function** as an even more general concept to describe a system as the Laplace transform of the output signal divided by the Laplace transform of the input signal. The transfer function can also be seen as the Laplace transform of the impulse response function.The frequency response function could be derived from the transfer function by letting $s = j\omega$. We found that the transfer function of the RC circuit has a pole at the location $-1/RC$ (on the negative real axis of the s plane). We also found that the (causal) impulse response of a system with a single pole is proportional to an exponential function $e^{s_p t}$ with s_p being the location of the pole. Therefore the causal system can only be **stable** if the pole is located within the left half plane of the s plane. Next, we introduced the step response function as yet another way to express the action of a filter. It was shown that the step response function and the impulse response function are closely related and can be obtained from each other by integration and differentiation, respectively.

We found a way to graphically determine the frequency response function given the pole position in the s plane. From analysing the frequency response function in a log-log plot, we derived the rule that a pole in the transfer function causes a change of the slope of the frequency response function at a frequency ω_c by 20 dB/decade with ω_c being the distance of the pole from the origin of the s plane. We finally approximated the **differential equation** of the RC circuit by its difference equation which could be solved iteratively.