General linear time invariant systems

Generalization of concepts

Rewrite the differential equation for the RC filter:

$$RC\dot{y}(t) + y(t) - x(t) = \alpha_1 \frac{d}{dt} y(t) + \alpha_0 y(t) + \beta_0 x(t) = 0$$

$\Longrightarrow$ RC filter special case of
Nth order LTI system

$$\sum_{k=0}^{N} \alpha_k \frac{d^k}{dt} y(t) + \sum_{k=0}^{L} \beta_k \frac{d^k}{dt} x(t) = 0$$

Table 3.1 Correspondences between the RC filter (1st order system) and a general Nth order LTI system. For the RC filter $a_1 = RC$, $a_0 = 1$, $b_0 = 1$.

Concept	RC filter	General system
Differential equation	$\alpha_1 \frac{d}{dt} y(t) + \alpha_0 y(t) + \beta_0 x(t) = 0$	$\sum_{k=0}^{N} \alpha_k \frac{d^k}{dt} y(t) + \sum_{k=0}^{L} \beta_k \frac{d^k}{dt} x(t) = 0$
Transfer function	$T(s) = \frac{-\beta_0}{\alpha_0 + \alpha_1 s}$	$T(s) = \frac{\beta_0 + \beta_1 s + \beta_2 s^2 + \ldots + \beta_L s^L}{\alpha_0 + \alpha_1 s + \alpha_2 s^2 + \ldots + \alpha_N s^N}$
Frequency response function	$T(j\omega) = \frac{-\beta_0}{\alpha_0 + \alpha_1 j\omega}$	$T(j\omega) = \frac{\beta_0 + \beta_1 (j\omega) + \beta_2 (j\omega)^2 + \ldots + \beta_L (j\omega)^L}{\alpha_0 + \alpha_1 (j\omega) + \alpha_2 (j\omega)^2 + \ldots + \alpha_N (j\omega)^N}$
Poles and zeroes	A single pole at $-1/a_1$,the root of the denominator polynomial $T(s) = \frac{-\beta_0}{\alpha_0 \cdot (s - s_{p1})}$ or $T(j\omega) = \frac{-\beta_0}{\alpha_0 \cdot (j\omega - s_{p1})}$	N poles at the roots of the denominator polynomial, L zeroes at the roots of the numerator polynomial $T(s) = \frac{-\beta_L \cdot \prod_{k=1}^{L} (s - s_{0k})}{\alpha_N \cdot \prod_{k=1}^{N} (s - s_{pk})}$ or $T(j\omega) = \frac{-\beta_L \cdot \prod_{k=1}^{L} (j\omega - s_{0k})}{\alpha_N \cdot \prod_{k=1}^{N} (j\omega - s_{pk})}$
Difference equation	$y(n) = -a_1 y(n-1) + b_0 x(n)$	$y(n) = \sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{L} b_k x(n-k)$

ROC for system with 1 pole

(ROC...region of convergence)

- ROC = half planes bounded by the pole
- ROC = right half plane yields rightsided IR
- ROC = left half plane yields leftsided IR

ROC for general LTI systems

ROC in general bands parallel to the imaginary axis without poles

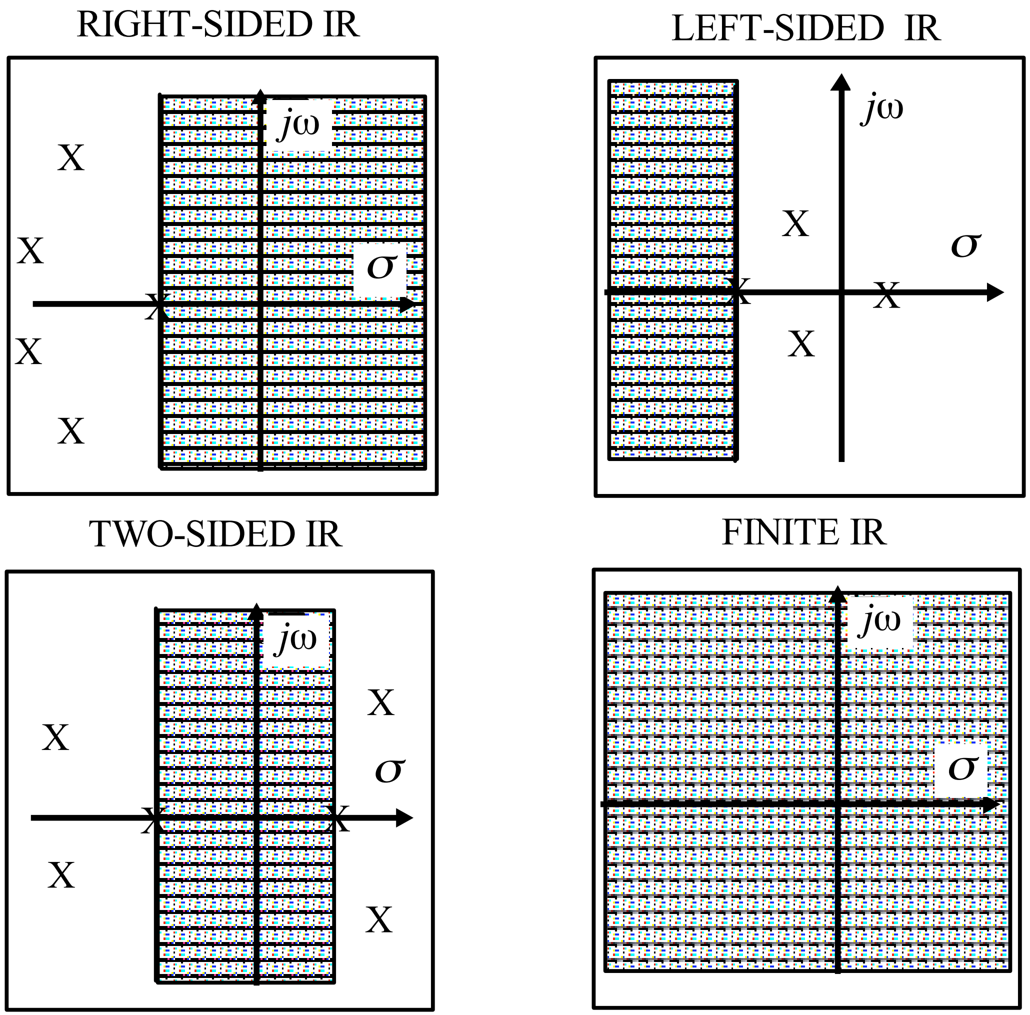


Fig. 3.1 Types of convergence regions for general LTI systems with existing Laplace transform and the corresponding types of stable (infinite) impulse response (IR) functions.

Problem 3.1

System with two poles. Consider three different cases:

a) Put both poles at -1.2566, 0.
b) Put one pole at location -1.2566, 0 and the other one at 1.2566, 0.
c) Put both poles at 1.2566, 0.

For the input signal, use a spike at the center position of the window (for DST an internal sampling frequency of 100Hz and a window length of 2048 points works well). What types of impulse response functions do you expect in each case? Will the frequency response functions be different? What changes do you expect for the frequency response functions with respect to Problem 2.3 (single pole at –1.2566,0)?

Consequences of transition from single pole to general N-th order system

- No major change in concept
- Zeros in addition to poles
- Can be treated in very similar way

Graphical estimation of the frequency response function

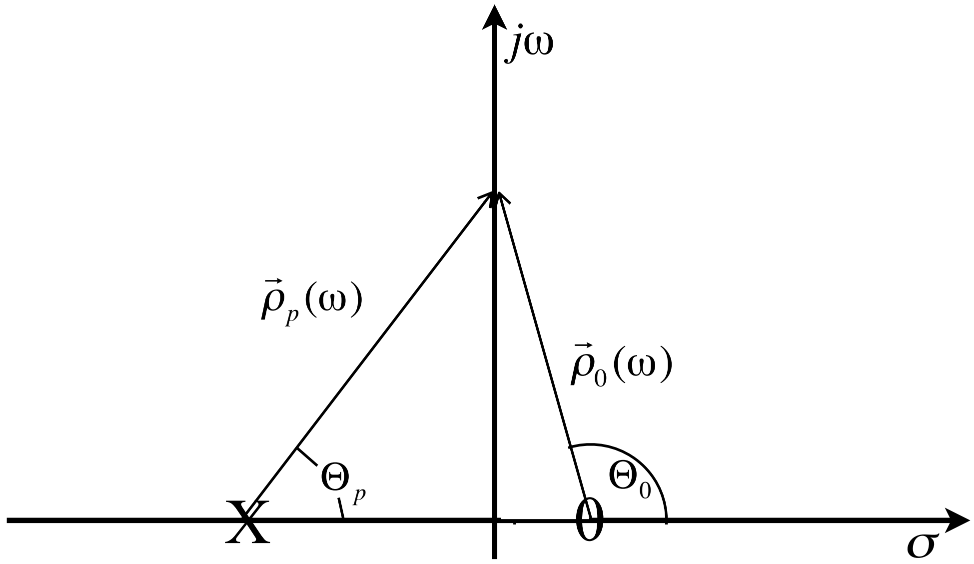


Fig. 3.2 Complex s plane representation of a system with a single pole and zero. The pole and zero locations are marked by an X, and a 0, respectively.

Transfer function: $T(s) = \dfrac{s - s_0}{s - s_p}$

s_0 and s_p: position of the zero and the pole, respectively.

Frequency response function: $(s = j\omega)$ $\quad T(j\omega) = \dfrac{j\omega - s_0}{j\omega - s_p}$

$j\omega - s_p \quad \rightarrow \quad$ vector $\vec{\rho}_p(\omega)$

$j\omega - s_0 \quad \rightarrow \quad$ vector $\vec{\rho}_0(\omega)$

$$T(j\omega) = \left|\rho_0(\omega)\right| e^{j\Theta_0} \frac{1}{\left|\rho_p(\omega)\right|} e^{-j\Theta_p}$$

Arbitrary LTI system

The amplitude part of the frequency response function of an arbitrary LTI system can be determined graphically by multiplying the lengths of the vectors from the zero locations in the S plane to the point $j\omega$ on the imaginary axis divided by the product of all lengths of vectors from pole locations to the point $j\omega$ on the imaginary axis. Likewise, to determine the phase part, the phase angles for the vectors from the zero locations in the S plane to the point $j\omega$ on the imaginary axis have to be added together. Then, the phase angles of all the vectors from pole locations to the point $j\omega$ on the imaginary axis have to be subtracted.

Problem 3.2

Use the argument given above to determine the frequency response for a system with a single pole at –1.2566,0 if you add a zero at position 1.2566, 0?

The phase properties of general LTI system

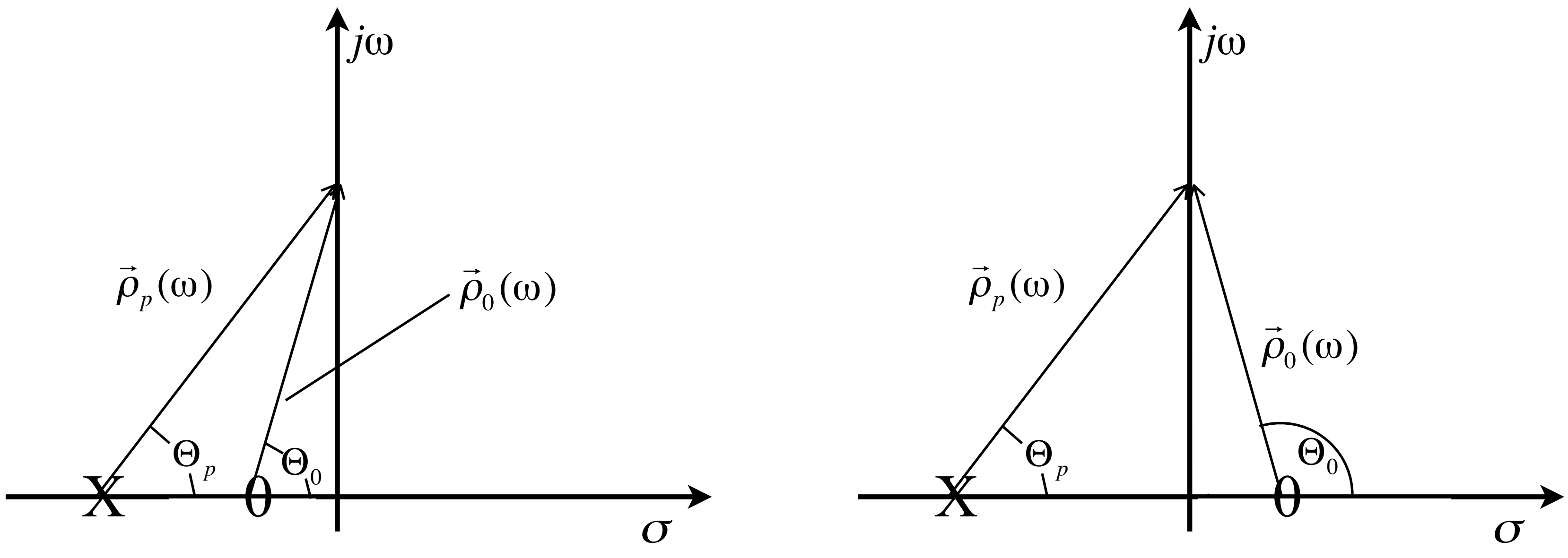


Fig. 3.3 Complex s plane representation of two systems with a single pole and zero. In b) the zero is at the same distance from the origin as in a) The pole and zero locations are marked by an X, and a 0, respectively.

How does the amplitude response differ?

How do the phase properties differ?

Minimum/maximum phase

A **causal stable** system (no poles in the right half plane) is minimum phase provided it has no zeroes in the right hand plane.

It is maximum phase if it has all its zeroes in the right hand plane.

minimum phase systems: nice properties !

Other phase properties

- mixed phase: neither minimum nor maximum phase
- linear phase: constant time shift
- zero phase: no phase distortion

Problem 3.3

How can the following two statements be proven for a general LTI system? a) If a system is minimum phase it will always have a stable and causal inverse filter. b) Any mixed phase system can be seen as a convolution of a minimum phase system and an allpass filter, which only changes the phase response but leaves the amplitude response as is.

Problem 3.4

How can we change the two-sided impulse response from Problem 3.1b (one pole at –1.2566,0 and another one at 1.2566,0) into a right-sided one without changing the amplitude response? Keyword: allpass filter.

The interpretation of the frequency response function

A single pole in the transfer function causes the slope of the amplitude frequency response function to decrease by 20 dB/decade (6 dB/octave).

How about a zero?

A single zero causes an increase of the slope by the same amount .

The transition in either case occurs at corner frequencies which are equal to the modulus of the pole/zero position.

General rule

Each pole in the transfer function causes the slope of the amplitude frequency response function to decrease by 20 dB/decade (6 dB/octave).

Each zero causes an increase of the slope by the same amount .

The transition in either case occurs at corner frequencies which are equal to the moduli of the pole/zero positions.

Problem 3.5

Consider a system with a pole and a zero on the real axis of the s plane. Let the pole position be (-6.28318, 0), and the zero position (.628318,0). What is the contribution of the zero to the frequency response function?

An internal sampling frequency of 20 Hz is recommended in DST

One more point...

• So far only singularities on the real axis. What happens for singularities away from real axis?

Are we allowed to do this?

Yes, but: Real systems require conjugate complex singularities.

Problem 3.6

Move the pole position of a double pole at (-1.2566,0) in steps of 15^o (up to 75^o and 85^o) along the circle with the radius corresponding to the corner frequency (ω_p) of the poles. Calculate impulse response functions and the amplitude portions of the frequency response functions.

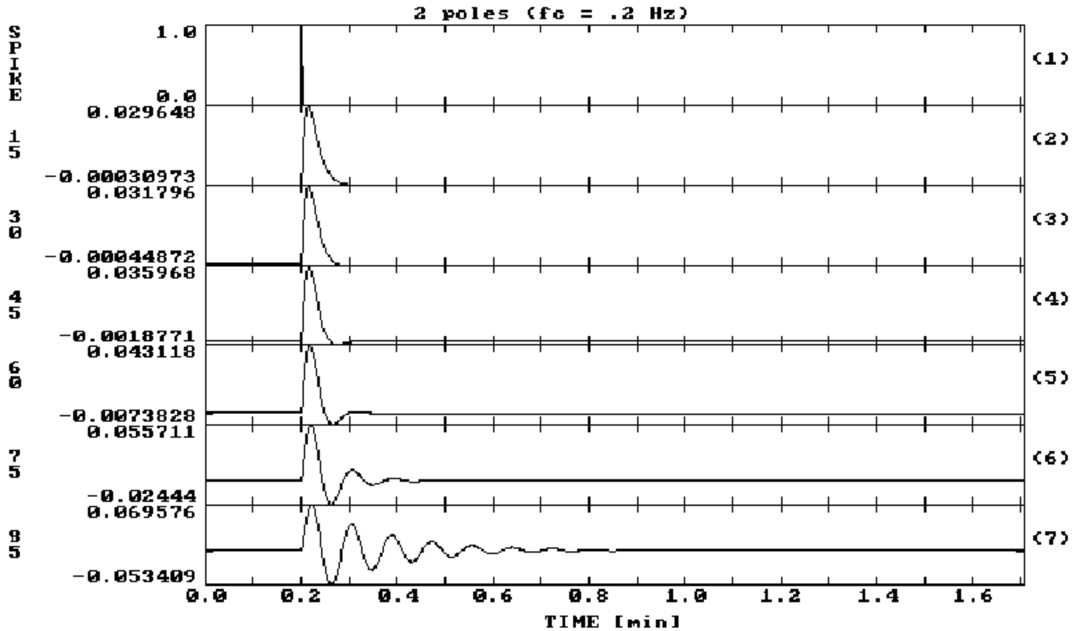


Fig. 3.5 Impulse responses for a system with a conjugate complex pole for the pole positions at different angles with the real axis of the s plane (15, 30, 45, 60, 75, 85°).

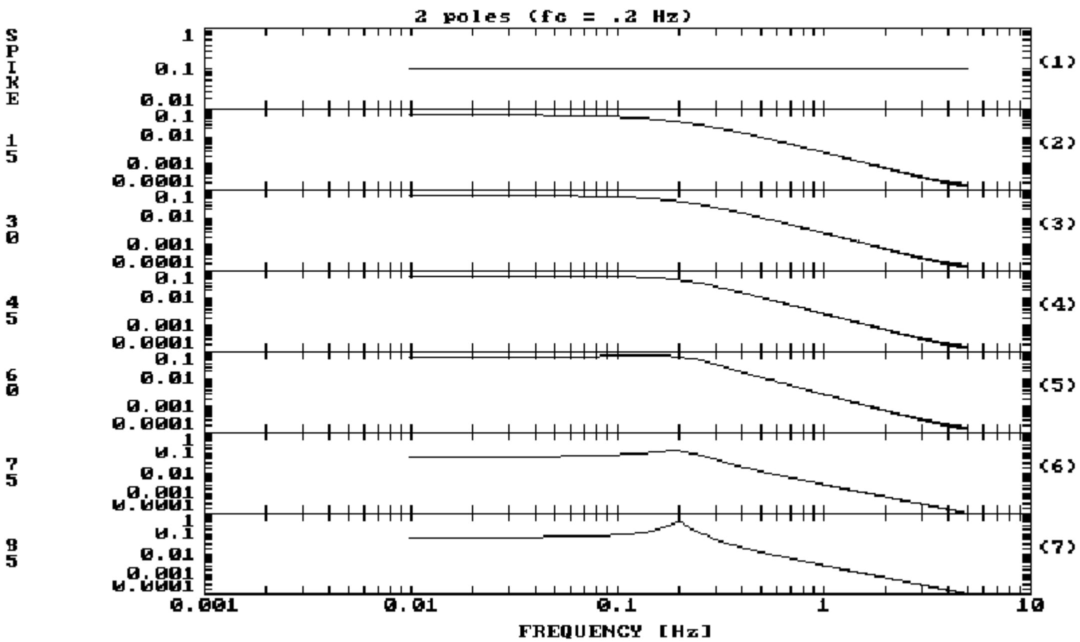
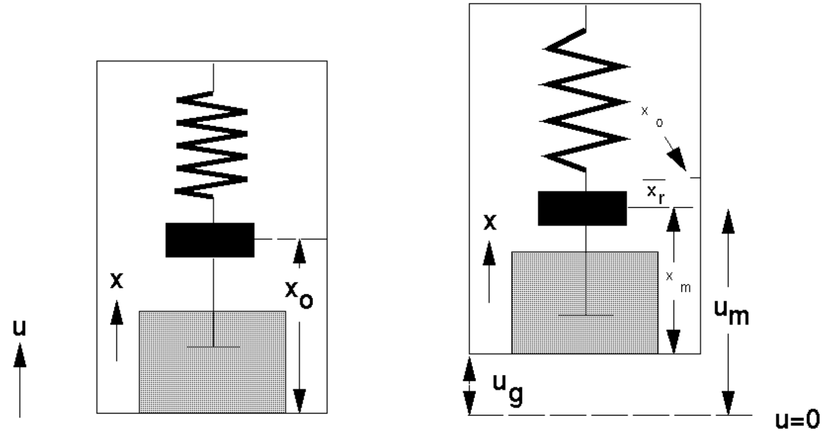


Fig. 3.6 Frequency response functions (amplitude) corresponding to Fig. 3.5.

Problem 3.7

Use the pole-zero approach to design a notch filter suppressing unwanted frequencies at 6.25 Hz. What kind of singularities do we need? How can we make use of the result of Problem 3.6?

Seismometer



- *The inertia of the mass —* $$f_i = -m\ddot{u}_m(t)$$

- *The spring —* $$f_{sp} = -kx_r(t)$$

$$k = \text{spring constant (strength)}$$

- *The dashpot —* $$f_f = -D\dot{x}_m(t)$$

$$D = \text{friction coefficient}$$

Seismometer Transfer function

Laplace transform of $\ddot{x}_r(t) + 2\varepsilon\dot{x}_r(t) + \omega_0^2 x_r(t) = -\ddot{u}_g(t)$

$$\Rightarrow \quad s^2 X_r(s) + 2\varepsilon s X_r(s) + \omega_0^2 X_r(s) = -s^2 U_g(s)$$

and $\quad (s^2 + 2\varepsilon s + \omega_0^2) X_r(s) = -s^2 U_g(s)$

Transfer function: $\quad T(s) = \dfrac{X_r(s)}{U_g(s)} = \dfrac{-s^2}{s^2 + 2\varepsilon s + \omega_0^2}$

Since a quadratic equation $x^2 + bx + c = 0$ has the roots

$$x_{1,2} = -\frac{b}{2} \pm \sqrt{\frac{b^2}{4} - c}$$

pole positions $p_{1,2}$: $$p_{1,2} = -\varepsilon \pm \sqrt{\varepsilon^2 - \omega_0^2}$$

$$= -h\omega_0 \pm \omega_0\sqrt{h^2 - 1}$$

$$= -\left(h \pm \sqrt{h^2 - 1}\right)\omega_0$$

For the underdamped case ($h < 1$) the pole position becomes

$$p_{1,2} = -\left(h \pm j\sqrt{1 - h^2}\right)\omega_0$$

with the pole distance from the origin

$$\left|p_{1,2}\right| = \left|\left(h \pm j\sqrt{1 - h^2}\right)\right| \cdot \left|\omega_0\right| = \sqrt{h^2 + (1 - h^2)} \cdot \left|\omega_0\right| = \left|\omega_0\right|$$

Therefore, the poles of an underdamped seismometer are located in the left half of the s plane in a distance of $|\omega_0|$ from the the origin. The quantity $h\,|\omega_0|$ gives the distance from the imaginary axis.

Problem 3.8

From the shape of the frequency response function in the figure determine the poles and zeroes of the corresponding transfer function.

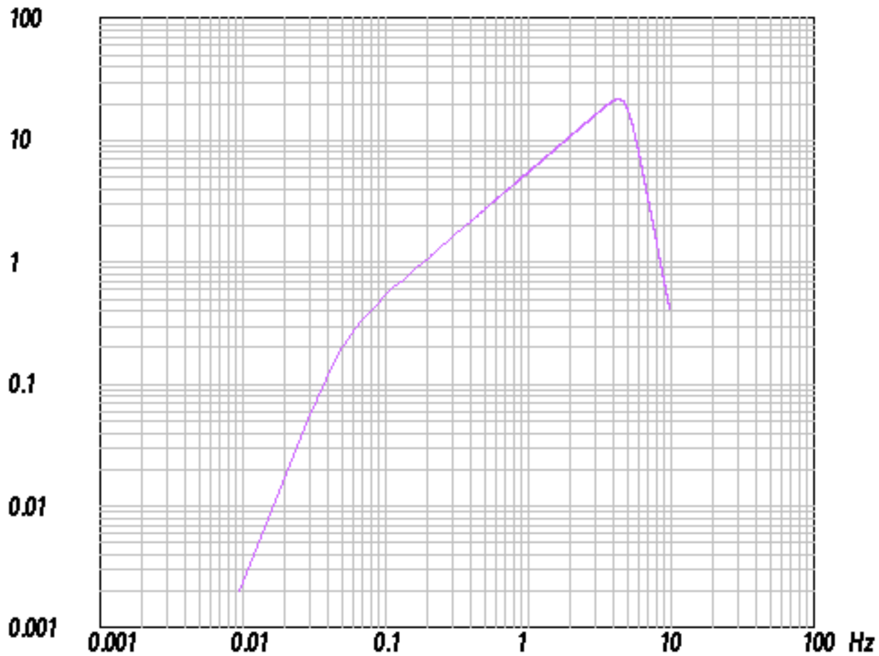


Fig. 3.4 Frequency response function (amplitude) with an unknown pole - zero distribution.