Sampling and A/D conversion

= 2 step procedure

• *Sampling or discretization* — Taking discrete samples of a continuous data stream. The data may still be in analog representation after the sampling process.

• *Analog to digital conversion (quantization and coding)* — For voltage signals, this steps normally occurs in an electronic device which is called ADC, 'analog to digital converter'.

The sampling process

$1/T = f_{dig}$, is called the *sampling frequency* or the *digitization frequency*.

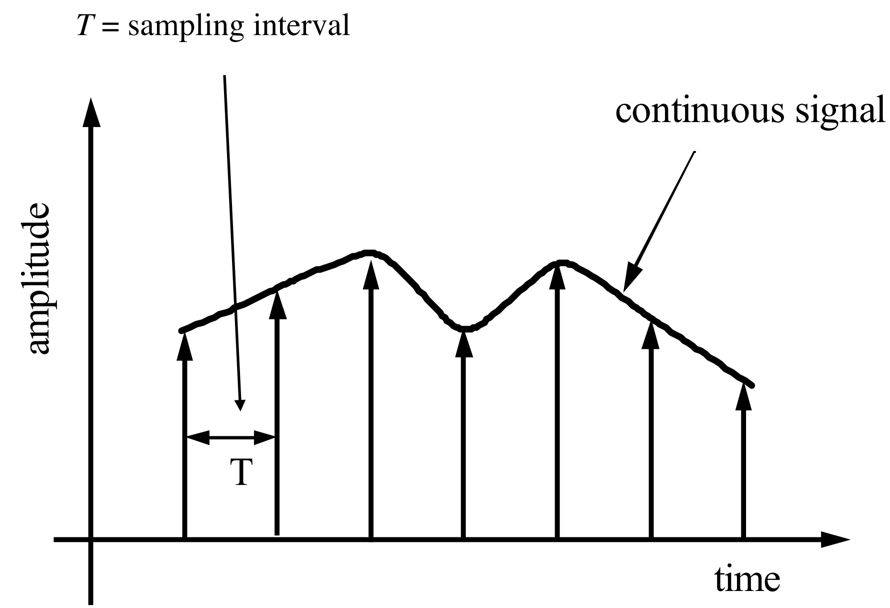


Fig. 5.1 Sketch of the discretization (sampling) process. The vertical arrows show the locations and the values of the samples. *T* denotes the sampling interval.

I) Discrete Signals

1.Discretization - Sampling

A continuous Signal Function: $x_a(t)$ taken at specific time steps T_s results in:

$$x[n] = x_a(nT_s);$$

T_s = sampling interval; $f_s = \frac{1}{T_s}$ = sampling rate or sampling frequency

Note! The amplitude values are still $x_a(t) \in \boldsymbol{R}$!!!!

Different mathematical notation using the 1-Impulse:

$$\delta(t) = \begin{bmatrix} 1 \text{ if t=0} \\ 0 \quad \text{sonst} \end{bmatrix};$$

and using a series of such “1-Impulses” describes the sampling:

$$\delta_{T_s}(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$

This results in:

$$x[n] = x_a(t)\delta_{T_s}(t)$$

The Sampling Theorem

In order to describe a continuous signal or function complete and unique using amplitude values taken at discrete times T_s, the sampled signal **MUST NOT HAVE** energy above a certain frequency $\frac{f_s}{2} = \frac{1}{2T_s}$. This frequency is also called **Nyquist-Frequency.**

The corresponding continuous signal $x_a(t)$ could be reconstructed using a linear combination of the discrete function weighted by a function $\mathrm{sinc}(t) = \frac{\sin(t)}{t}$:

$$x_a(t) = \sum_{n=-\infty}^{\infty} x[n]\,\mathrm{sinc}(\pi f_s(t - nT_s))$$

2.Sampling - Fourier Transform

Definition:

$$X_a(j\omega) = \int_{-\infty}^{\infty} x_a(t)e^{-j\omega t}\ dt \Leftrightarrow x_a(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_a(j\omega)e^{j\omega t}\ d\omega;$$

Strictly only valid if:

the function is absolute integrable: $\int |x_a(t)|\ dt \leq c < \infty$

Latter point is not always the case in Geophysics!

Important Properties of the Fourier Transform

Time domain	Frequency domain
x(t) real	$X(-j\omega) = [X(j\omega)]^*$
x(t) imaginary	$X(-j\omega) = -[X(j\omega)]^*$
$x(t) = x(-t)$ (x(t) even)	$X(-j\omega) = X(j\omega)$ ($X(j\omega)$ even)
$x(t) = - x(-t)$ (x(t) odd)	$X(-j\omega) = -X(j\omega)$ ($X(j\omega)$ odd)
x(t) real and even	$X(j\omega)$ real and even
x(t) real and odd	$X(j\omega)$ imaginary and odd
x(t) imaginary and even	$X(j\omega)$ imaginary and even
x(t) imaginary and odd	$X(j\omega)$ real and odd
	* complex conjugate

Time domain	**Frequency domain**
multiplication: $x_a(t)h_a(t)$	convolution: $\frac{1}{2\pi}X_a(j\omega) \bullet H_a(j\omega) = \frac{1}{2\pi}\int X_a(j\Omega)H_a(j\omega - j\Omega)\ d\Omega$
convolution: $x_a(t) \bullet h_a(t)$	multiplication: $X_a(j\omega)H_a(j\omega)$
differentiation: $\frac{dx(t)}{dt}$	multiplication: $j\omega \cdot X(j\omega)$
integration: $\int x_a(t)\ dt$	multiplication: $\frac{1}{j\omega}X_a(j\omega)$
$x_a(t-a)$	$X_a(j\omega)e^{-j\omega a}$ for $a > 0$
	* complex conjugate

Parsevals-Theorem:

$$\int_{-\infty}^{\infty} |x_a(t)|^2 \; dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X_a(j\omega)|^2 \; d\omega ;$$

Back to sampling process:

$$FT\left(\sum_{n=-\infty}^{\infty} \delta(t - nT_s) \right) = \frac{2\pi}{T_s} \sum_{k=-\infty}^{\infty} \delta\left(\omega - k\frac{2\pi}{T_s} \right) = \Delta_T(j\omega)$$

using

$\omega_s = 2\pi f_s = \dfrac{2\pi}{T_s}$ this results in $\Delta_T(j\omega) = \dfrac{2\pi}{T_s} \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s)$

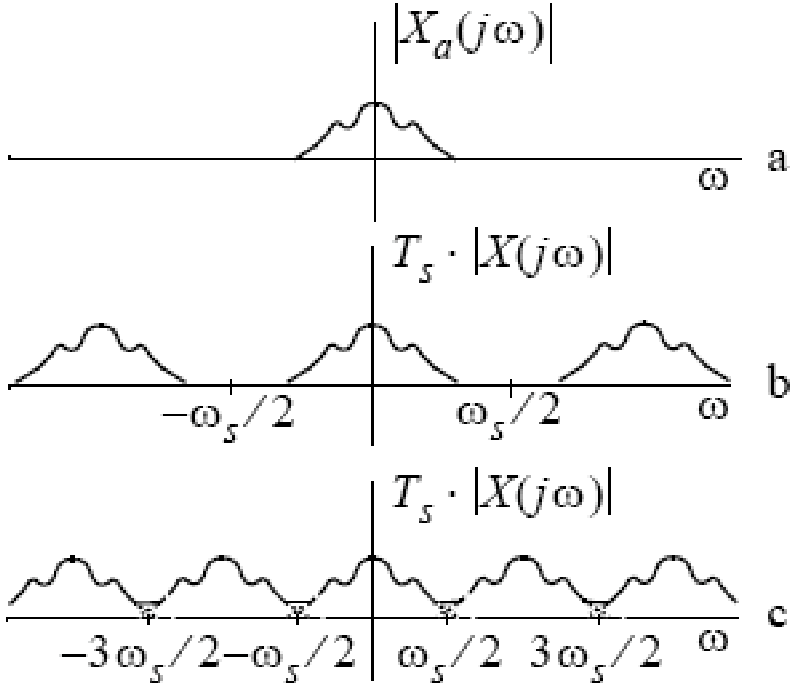
FT of a sampled signal can be represented by a convolution of $FT\{\Delta_T\}$ and $FT\{X_a(j\omega)\}$

$$X(j\omega) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} X_a(j\omega - kj\omega_s)$$

What the heck does that mean????

The sampled signal $x[n]$ will be periodic in frequency ω_s (sampling frequency). It follows that the continuous signal $x(t)$ can be reconstructed using only one period. Only valid if the sampling theorem is not violated and no energy above $\frac{\omega_s}{2}$ is present in the signal $x(t)$:

$$X_a(j\omega) = T_s X(j\omega);$$

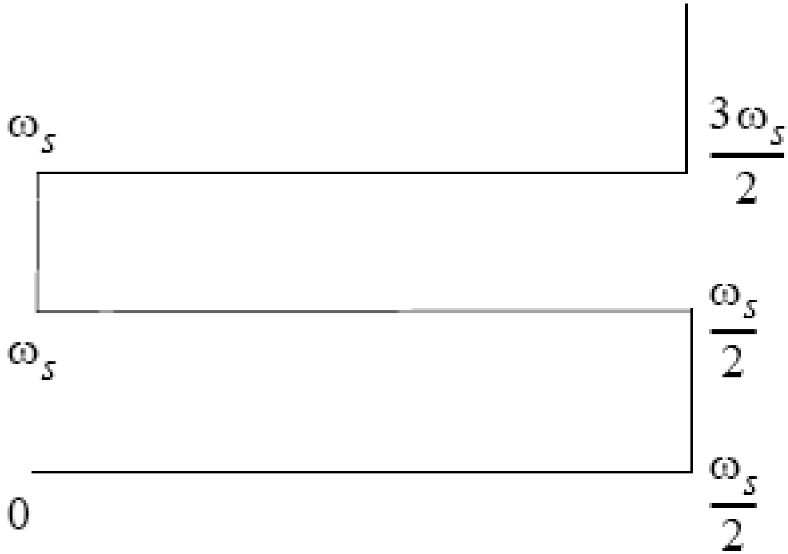


a) FT of analog signal
b) FT of discrete signal (sampling theorem complied)
c) FT of discrete signal (sampling theorem violated)

The sampling theorem **MUST** be applied **BEFORE** the sampling process. Therefore an analog lowpass filter must be applied before sampling - regardless which sampling frequency is used. The corner frequency (!) of that filter should satisfy:

$$f_c = 0.4 \cdot f_s.$$

Consequence of violation - **ALIASING**:



Problem 1:
Assume we are sampling with 125 Hz without an analog lowpass. Estimate the alias frequencies of noise signals at 70 Hz, 120 Hz and 300 Hz!

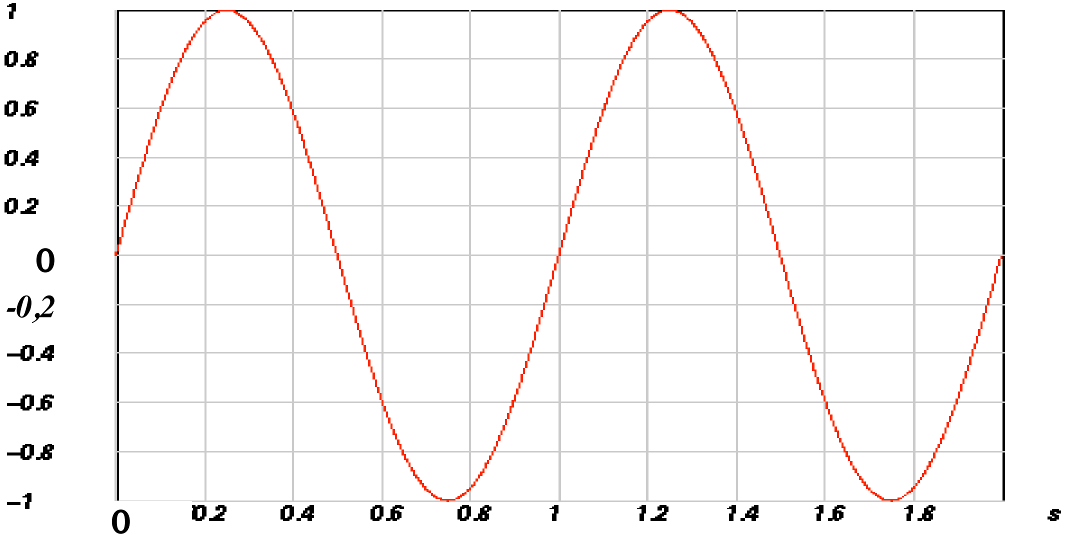


Fig. 5.2 Input signal for the simulation of the discretization process. The signal frequency is 1 Hz.

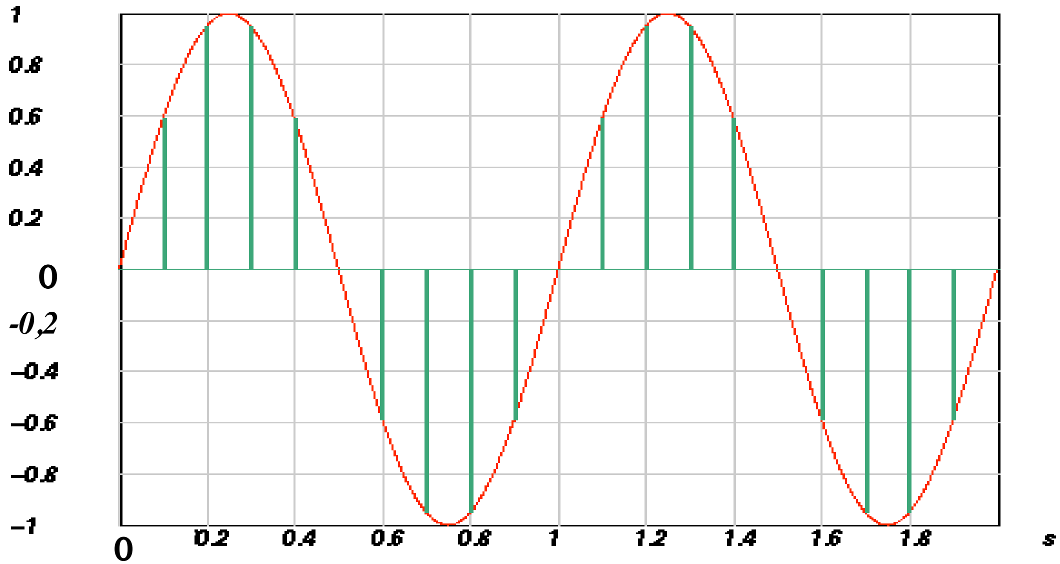


Fig. 5.3 Discretizing the data trace of Fig. 5.2 using a discretization frequency of 10 Hz. The vertical bars show the locations and the values of the function at the sampled times.

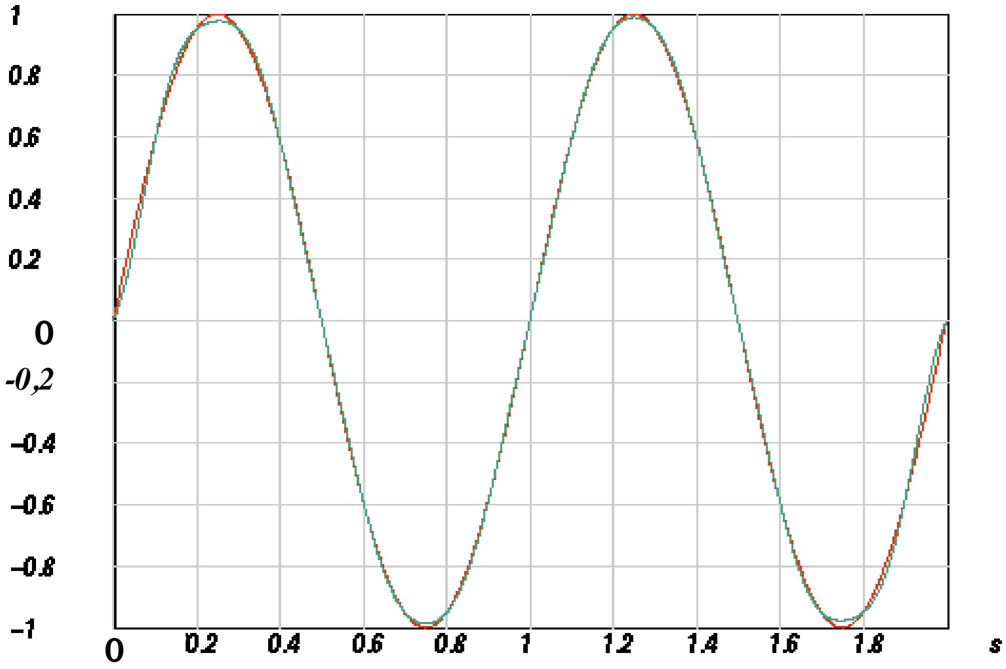


Fig. 5.4 Original and reconstructed trace of Fig. 5.2 (after discretizing all of them with 10 Hz prior to reconstruction).

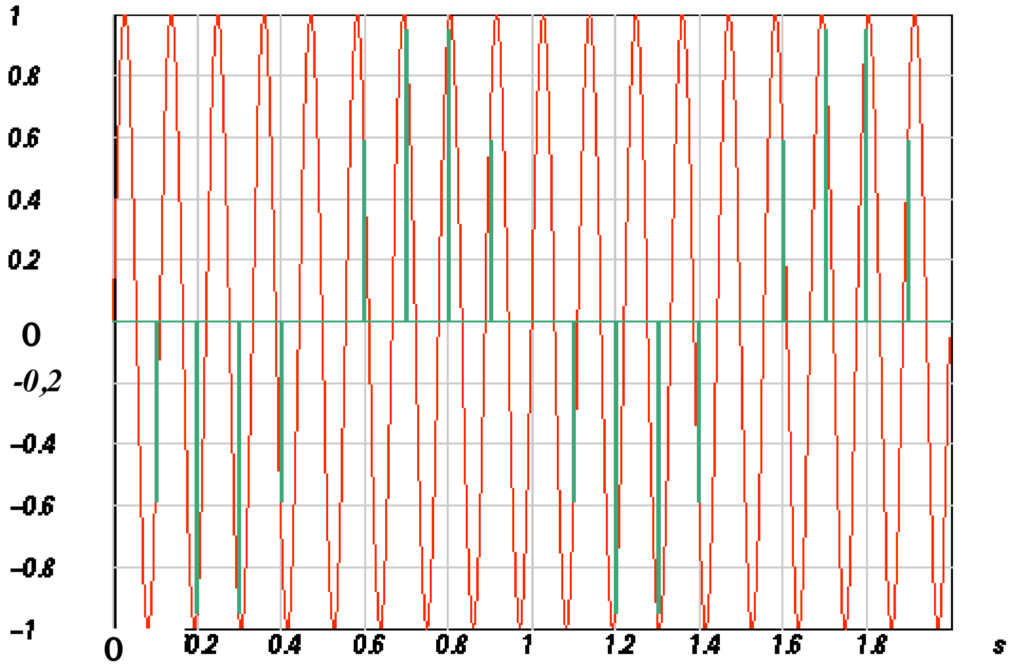


Fig. 5.6 Discretizing a sinusoidal signal with a signal frequency of 9 Hz and discretization frequency of 10 Hz. The vertical bars show the locations and the values of the function at the sampled times

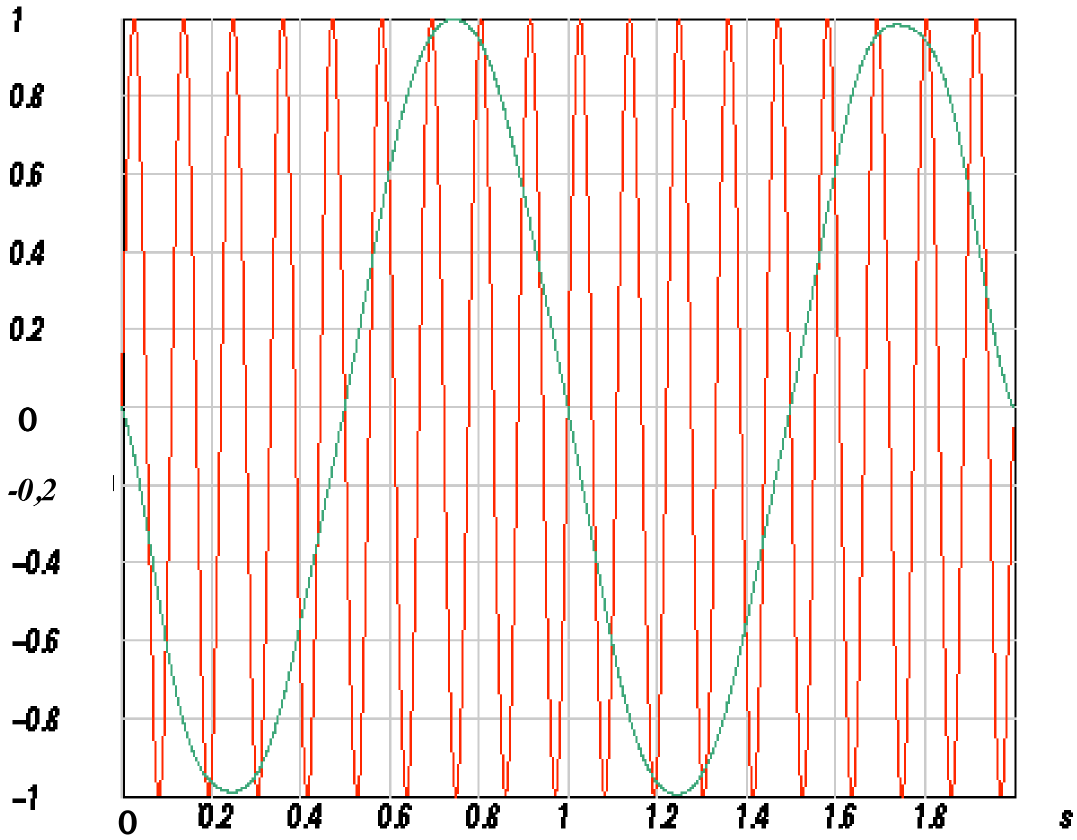


Fig. 5.5 Original and reconstructed sinusoidal signal with a signal frequency of 9 Hz (discretization frequency 10 Hz).

3.A/D-Conversion

Decimal system:

$$x_{(10)} = \sum_i d_i^{(10)} 10^i;$$

Example:

$$1024_{(10)} = 4 \cdot 10^0 + 2 \cdot 10^1 + 0 \cdot 10^2 + 1 \cdot 10^3$$

LSB MSB

Binary system:

$$x_{(2)} = \sum d_i^{(2)} 2^i$$

Example:

$512_{(10)} = 0 \cdot 2^0 + \ldots + 0 \cdot 2^8 + 1 \cdot 2^9$ represents “Little Endian”

LSB MSB 000000001

A 16 bit A/D-converter could represent in principle 2^{16} output states in its maximum (values between 0 -(2^n-1) are possible).

The LSB (least significant bit) or smallest step width of the A/D-converter (resolution) is defined by:

$$LSB = \frac{\text{Maximale Voltage}}{2^n} = Q\,.$$

As the resolution is directly dependent on the number of bits, a n-bit A/D-converter has "n-bit" resolution. Unfortunately, there is no rule, which would specify a "critical" number of "must have" bits. It is simply like that: if we have more bits we will decrease the noise added to the signal

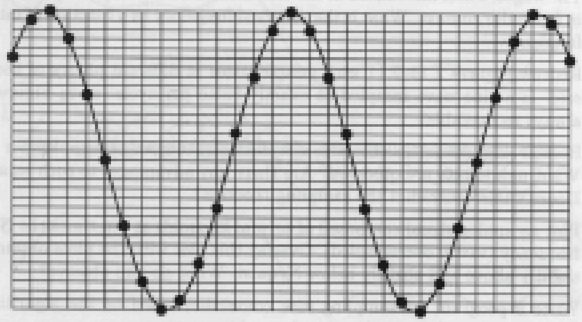


Figure 2.20: Conversion of an analog signal into a corresponding digital one involves quantizing both axes, sampling time and digitizing signal value. In the figure we see the original analog signal overlaid with the sampled time and digitized signal value grid. The resulting digital signal is depicted by the dots.

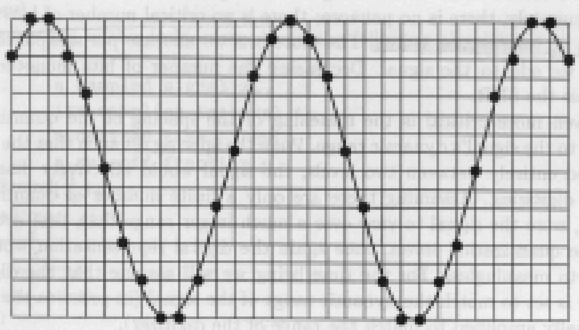


Figure 2.21: Conversion of an analog signal into the corresponding digital one with fewer digitizing levels. As in the previous figure the original analog signal has been overlaid with the sampled time and digitized signal value grid. However, here only 17 levels (about four bits) are used to represent the signal.

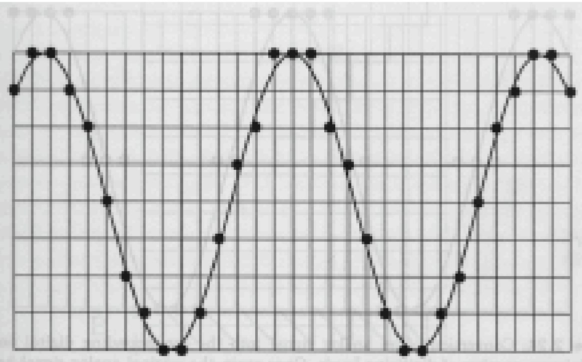


Figure 2.22: Conversion of an analog signal into the corresponding digital one with fewer digitizing levels. Once again the original analog signal has been overlaid with the sampled time and digitized signal value grid. Here only nine levels (a little more than three bits) are used to represent the signal.

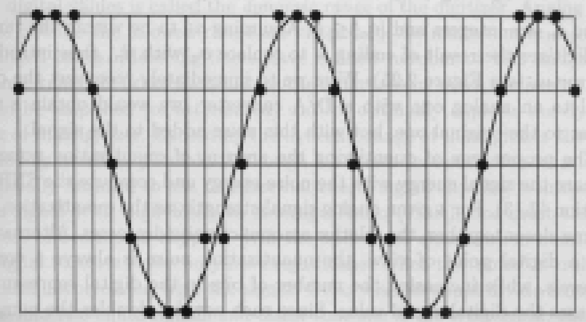


Figure 2.23: Conversion of an analog signal into the corresponding digital one with fewer digitizing levels. Once again the original analog signal has been overlaid with the sampled time and digitized signal value grid. Here only five levels (about two bits) are used to represent the signal.

An equivalent important parameter of A/D conversion is the so called dynamic range:

$$D = 20\log_{10}\left(\frac{A_{max}}{A_{min}}\right)$$

and therefore

$$D = 20\log_{10}(2^n - 1) \approx n\log_{10}(2) = n \cdot 6$$

Note: this definition intrinsically assumes proportionality to power (20*log) of the signal - **NOT** energy (10*log)!

16 bit A/D-converter: 90dB;
24 bit A/D-converter: 138 dB;

Be aware of the sign!

Exercise 1
Sampling and A/D Conversion

Problem 5.1

What is the general relationship between the **discretization frequency**, the signal frequency of a sinusoidal input signal ("**input frequency**") and the dominant frequency of the reconstructed signal ("**output frequency**")? Use DST to generate sinusoidal signals for an internal sampling frequency of 1024 Hz, a window length of 2048 points, and signal frequencies from 1 - 20 Hz in steps of 1 Hz. Discretize and reconstruct each signal using a discretization frequency of 10 Hz and note the dominant frequency and the maximum amplitude of the reconstructed signal. You can determine the dominant frequency of a signal in Hz easily by measuring the dominant signal period on the DST screen in seconds and taking the reciprocal value. From the table of input frequencies, output frequencies and output amplitudes, try to infer the rule for calculating the output frequency for a given signal frequency and a given digitization frequency. Hint: The so called Nyquist frequency (half of the discretization frequency) is also referred to as the *folding frequency*. Think of the "frequency band" as a foldable band which is folded at multiples of the Nyquist frequency. Mark the corresponding pairs of (input frequency, alias frequency) on this band. It may help to actually cut out a paper band and folding it.

Problem 5.2

What is the highest frequency which can be reconstructed correctly using a ‘discretization frequency’ of 10 Hz?

Problem 5.3

Problem 5.3 What would be the alias frequency for an input signal of 18.5 Hz and a discretization frequency of 10 Hz?

Sampling theorem

For a continuous time signal to be uniquely represented by samples taken at a sampling frequency of f_{dig}, (every $1/f_{dig}$ time interval), no energy must be present in the signal at and above the frequency $f_{dig}/2$. $f_{dig}/2$ is commonly called the *Nyquist frequency* (e.g. Mitra and Kaiser, 1993). Signal components with energy above the Nyquist frequency will be mapped by the sampling process onto the so called alias frequencies within the frequency band of 0 to Nyquist frequency. This effect is called the *alias* effect.

The Discrete Fourier Transform (DFT)

DFT is itself a finite length sequence. For a finite discrete-time sequence $x[nT]$ of length N, the DFT can be defined as:

$$\tilde{X}\left[k\right] = \frac{1}{N}\sum_{n=0}^{N-1} x\left[nT\right]e^{-j2\pi kn/N}$$

The set of sampled values can be recovered from the DFT by means of the inverse DFT, given by:

$$\tilde{x}\left[nT\right] = \sum_{k=0}^{N-1} \tilde{X}\left[k\right]e^{j2\pi kn/N}$$

• The inverse DFT yields the periodic sequence $\tilde{x}[nT]$ from which $x[nT]$ can be obtained by considering only a single period.

• Length of the sequence $x[nT]$ == integer power of 2 => FFT (fast FT).

• Except for the scaling factor of 1/ T the DFT it is equivalent to the Fourier series representation of the infinite periodic sequence $\tilde{x}[nT]$ which is made up by periodic extension of the given finite sequence $x[nT]$.

$$\tilde{X}[k] = c_k T$$

Here $\tilde{x}[nT]$ is the (infinite) periodic sequence constructed from $x[nT]$ by periodic continuation.

Notes

• In the context of the DFT always think in terms of $\tilde{x}[nT]$, the periodic extension of $x[nT]$ and consider $x[nT]$ as just one period of $\tilde{x}[nT]$

• The DFT is only defined for discrete radian frequencies w_k, which are related to the total number of points N, and the sampling interval T by:

$$\omega_k = k \cdot \frac{2\pi}{TN} \qquad \text{for k} = 0, 1, \ldots - \ldots, N-1$$

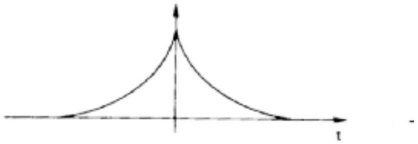
or in terms of frequencies f_k

$$f_k = k \cdot \frac{1}{TN} = k \cdot \frac{f_{dig}}{N} \qquad \text{for k} = (0, 1, \ldots - \ldots, N-1)$$

with f_{dig} being the sampling frequency.

Properties of certain signal types and corresponding spectra

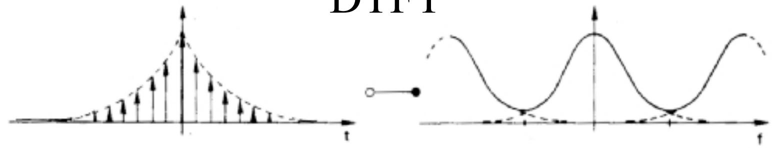
starting with:



- continous-time
- infinite signals

→

DTFT

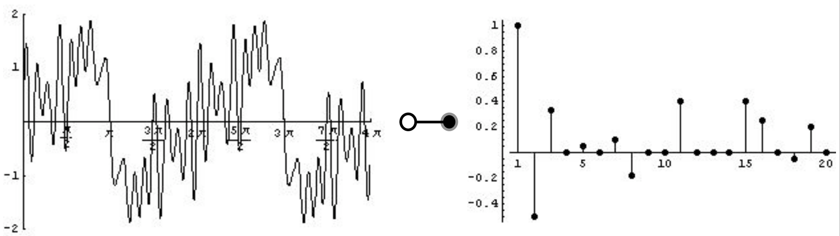


- discrete-time
- infinite signal

- periodic
- infinite spectrum

↓

CTFS

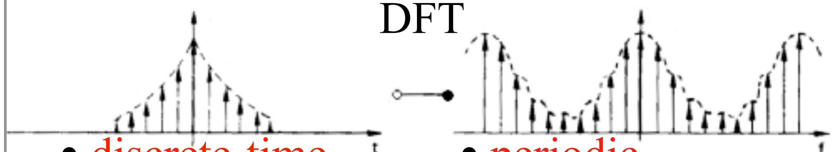


- continous-time
- periodic signal

- aperiodic
- discrete spectrum

→

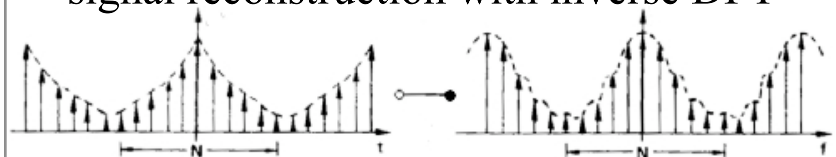
DFT



- discrete-time
- finite signal

- periodic
- discrete spectrum

signal reconstruction with inverse DFT



- discrete-time
- periodic signal

- periodic
- discrete spectrum

Numerical implementation

•For implementation, the DFT is most commonly defined for mere sequences of numbers and the notation $x[n]$ is used instead of $x[nT]$ (e.g. Strum and Kirk, 1988; Oppenheim and Schafer, 1989).

• Furthermore, the scaling factor $1/N$ is most commonly kept with the inverse transform. Therefore, a definition commonly found in software packages (e.g. IEEE Digital Signal Processing Committee, 1979) is:

$$\tilde{\tilde{X}}\left[k\right] = \sum_{n=0}^{N-1} x\left[n\right] e^{-j2\pi kn/N}$$

for the DFT and

$$\tilde{\tilde{x}}\left[n\right] = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}\left[k\right] e^{j2\pi kn/N}$$

for the inverse DFT. This directly leads to a common practical problem.

The z-transform and the discrete transfer function

Analog components of DAS: **analog transfer function *T*(*s*).**

Digital component: **discrete transfer function *T*(*z*).**

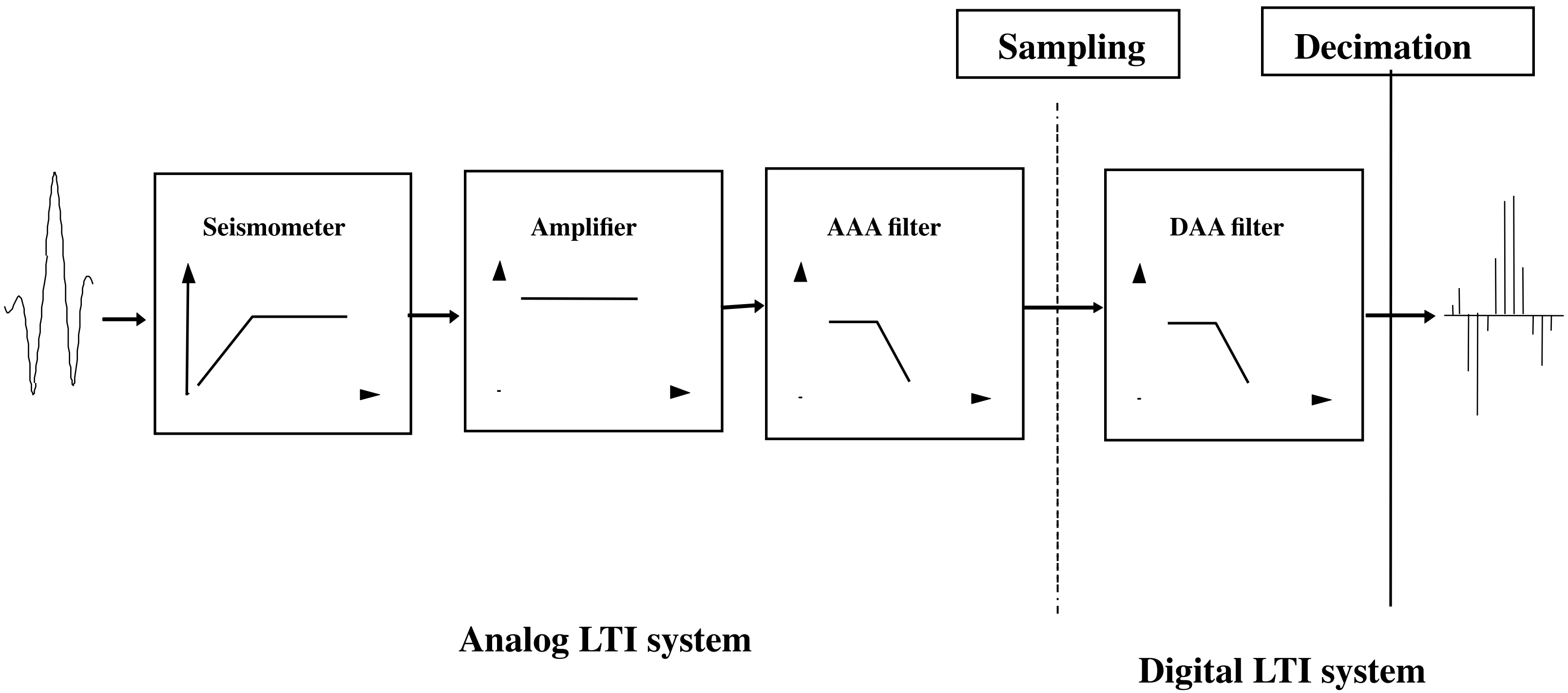
While *T*(*s*) was defined in terms of the Laplace transform, *T*(*z*) is defined in terms of its discrete counterpart, **the z-transform**.

The **bilateral z-transform** of a discrete sequence $x[n]$ is defined as

$$\mathbf{Z}\left\{x\left[n\right]\right\} = \sum_{n=-\infty}^{\infty} x\left[n\right] \cdot z^{-n} = X(z)$$

The z-transform transforms the sequence $x[n]$ into a function $X(z)$ with z being a continuous complex variable.

FIR - Filter Effects

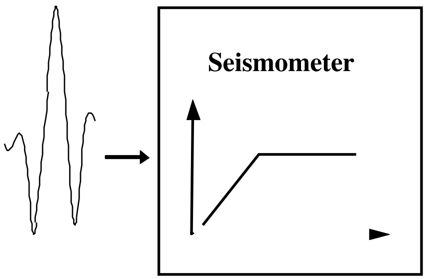


Why bothering?

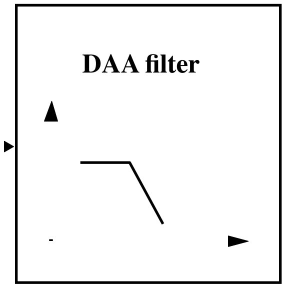
What is the reason for doing FIR filtering and decimating?

Nearly all seismic recorders use the oversampling technique to increase the resolution of recordings. In order to achieve an optimum valid frequency band, the filters are very steep.

Besides its advantages this also bears new problems.




```
#          << IRIS SEED Reader, Release 4.4 >>
#
#          ======== CHANNEL RESPONSE DATA ========
B050F03    Station:    RJOB
B050F16    Network:    BW
B052F03    Location:   ??
B052F04    Channel:    EHZ
B052F22    Start date: 2007,199
B052F23    End date:   No Ending Time
#          =======================================
#          +          +-------------------------------------------+          +
#          +          |  Response (Poles & Zeros),  RJOB ch EHZ   |          +
#          +          +-------------------------------------------+          +
#
B053F03    Transfer function type:          A [Laplace Transform (Rad/sec)]
B053F04    Stage sequence number:           1
B053F05    Response in units lookup:        M/S - Velocity in Meters per Second
B053F06    Response out units lookup:       V - Volts
B053F07    A0 normalization factor:         6.0077E+07
B053F08    Normalization frequency:         1
B053F09    Number of zeroes:                2
B053F14    Number of poles:                 5
#          Complex zeroes:
#            i  real          imag          real_error    imag_error
B053F10-13   0  0.000000E+00  0.000000E+00  0.000000E+00  0.000000E+00
B053F10-13   1  0.000000E+00  0.000000E+00  0.000000E+00  0.000000E+00
#          Complex poles:
#            i  real          imag          real_error    imag_error
B053F15-18   0 -3.700400E-02  3.701600E-02  0.000000E+00  0.000000E+00
B053F15-18   1 -3.700400E-02 -3.701600E-02  0.000000E+00  0.000000E+00
B053F15-18   2 -2.513300E+02  0.000000E+00  0.000000E+00  0.000000E+00
B053F15-18   3 -1.310400E+02 -4.672900E+02  0.000000E+00  0.000000E+00
B053F15-18   4 -1.310400E+02  4.672900E+02  0.000000E+00  0.000000E+00
```




```
#          +               +--------------------------------+               +
#          +               |  FIR response,  RJOB ch EHZ  |               +
#          +               +--------------------------------+               +
#
B061F03    Stage sequence number:              3
B061F05    Symmetry type:                     A
B061F06    Response in units lookup:           COUNTS - Digital Counts
B061F07    Response out units lookup:          COUNTS - Digital Counts
B061F08    Number of numerators:               96
#          Numerator coefficients:
#           i, coefficient
B061F09      0  3.767143E-09
B061F09      1  5.277283E-07
B061F09      2  2.184651E-06
B061F09      3 -5.639535E-06
B061F09      4 -1.233773E-06
B061F09      5  9.386712E-06
B061F09      6  4.859924E-06
B061F09      7 -1.644319E-05
…
#
#          +               +------------------------------+               +
#          +               |  Decimation,  RJOB ch EHZ  |               +
#          +               +------------------------------+               +
#
B057F03    Stage sequence number:              4
B057F04    Input sample rate:                 1.000000E+03
B057F05    Decimation factor:                  5
B057F06    Decimation offset:                 0
B057F07    Estimated delay (seconds):          1.490000E-01
B057F08    Correction applied (seconds):       0.000000E+00
```

Linear Difference Equation

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{l=0}^{M} b_l x[n-l]$$

Infinite Impulse Response: $a_k \neq 0$

Finite Impulse Response: $a_0 = 1$; $a_{k\neq 0} = 0$

- FIR filters :

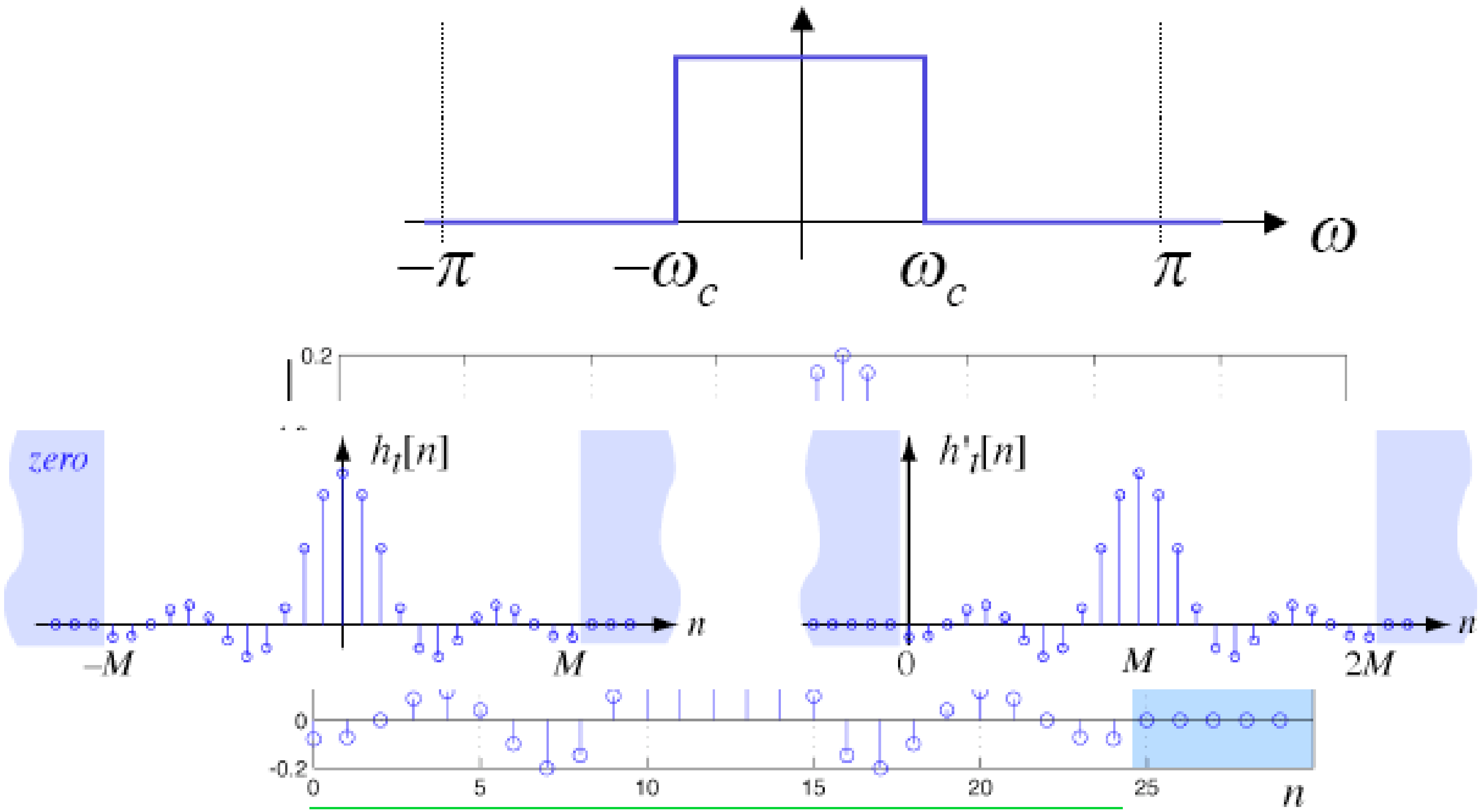
+ Always stable.
- Steep filters need many coefficients.
+ Both causal and noncausal filters can be implemented.
+ Filters with given specifications are easy to implement!

- IIR filters :

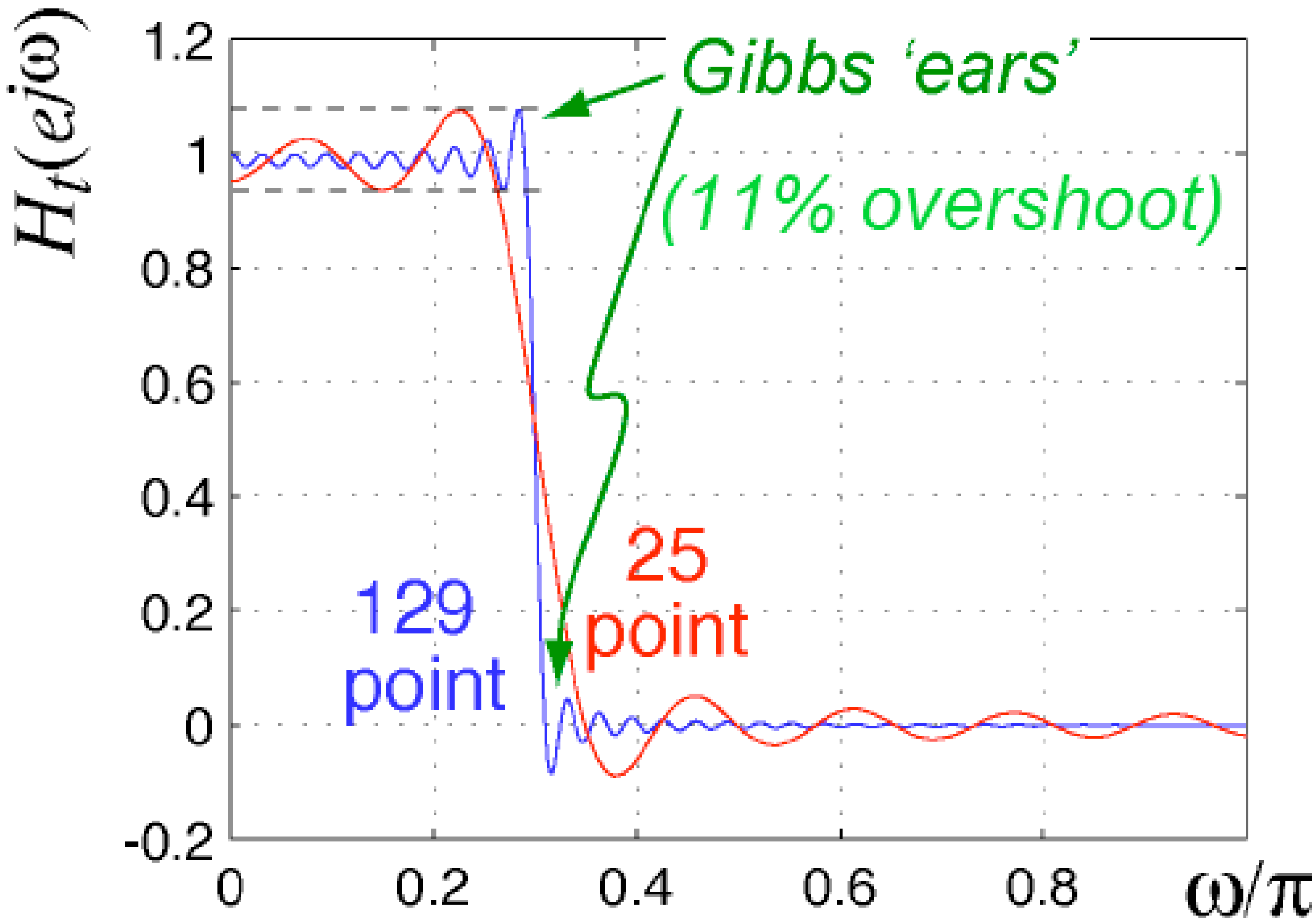
- Potentially unstable and subject to quantization errors.
+ Steep filters can easily be implemented with a few coefficients. Speed.
- Filters with given specifications are in general, difficult, if not impossible, to implement *exactly(!)*.

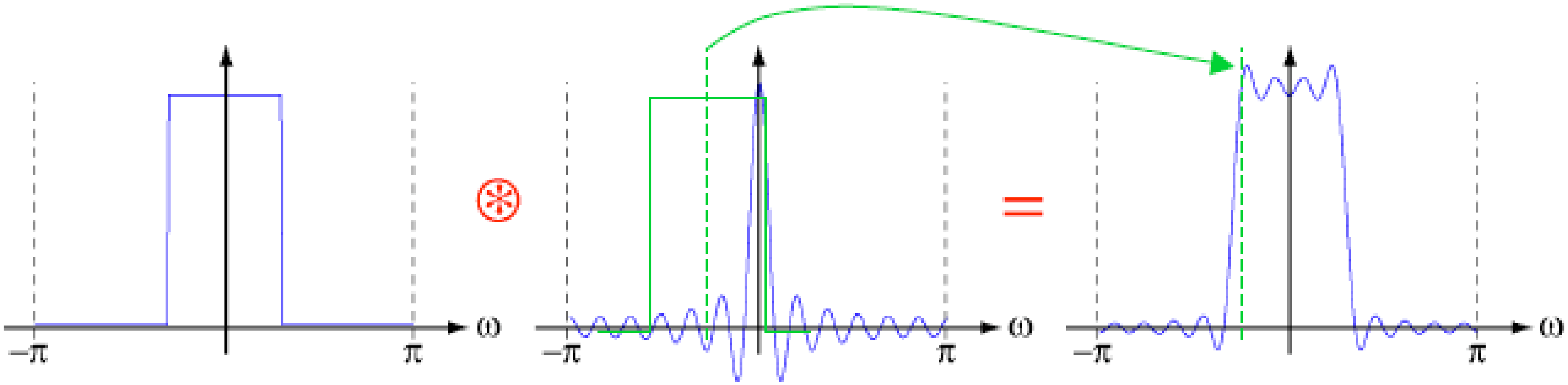
In SEED the impulse response of the decimation filters are given. But how to construct FIR filters?

Easiest way: inverse DFT with selected spectral shape and phase and truncate the (infinite) sequence to form a finite impulse response



Filter length vs. Steepnes





$H_d(e^{j\omega})$ $\circledast$ DTFT$\{w_R[n]\}$ $=$ $H_t(e^{j\omega})$

periodic sinc...

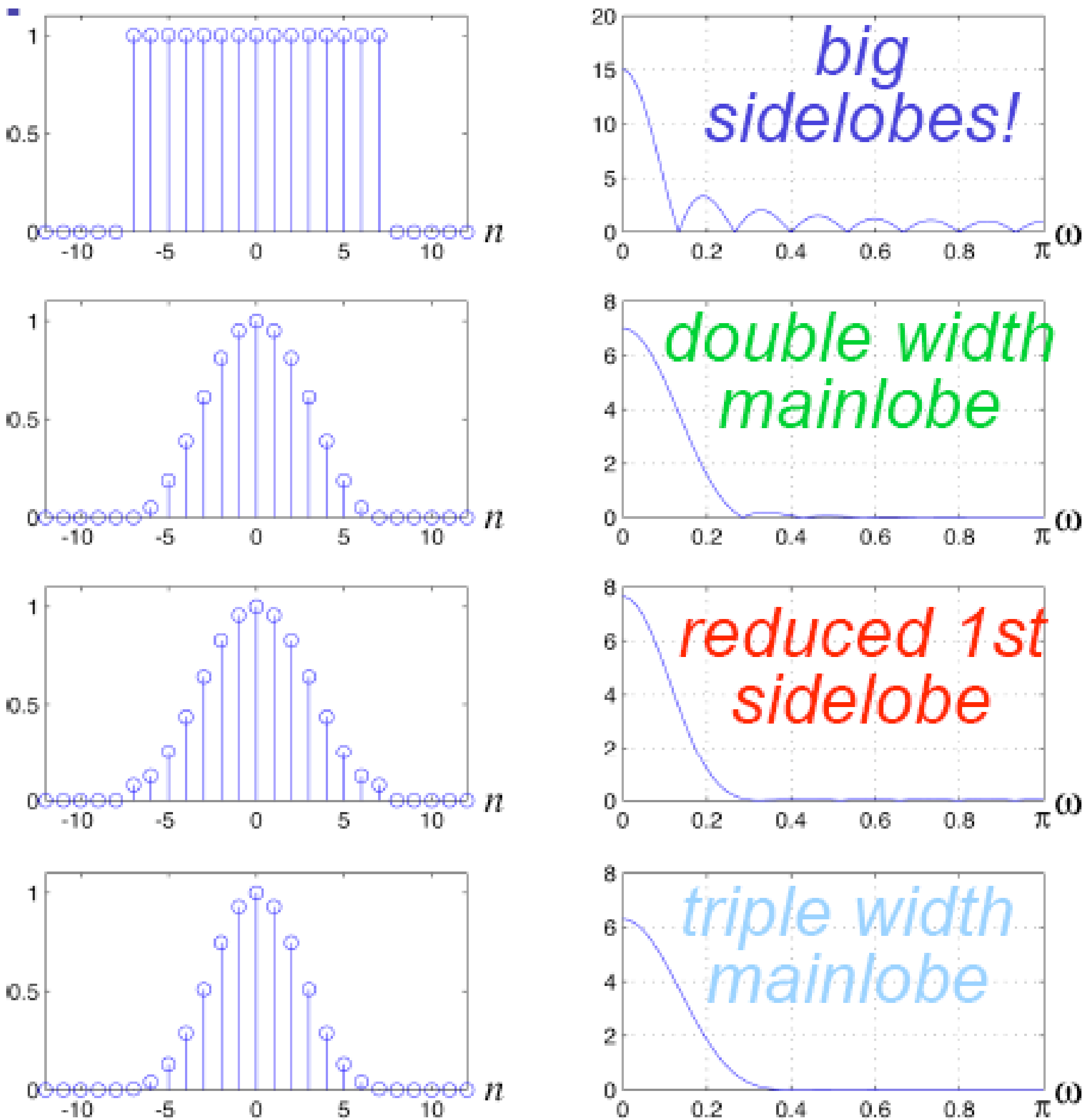
$$\frac{\sin Nx}{\sin x}$$

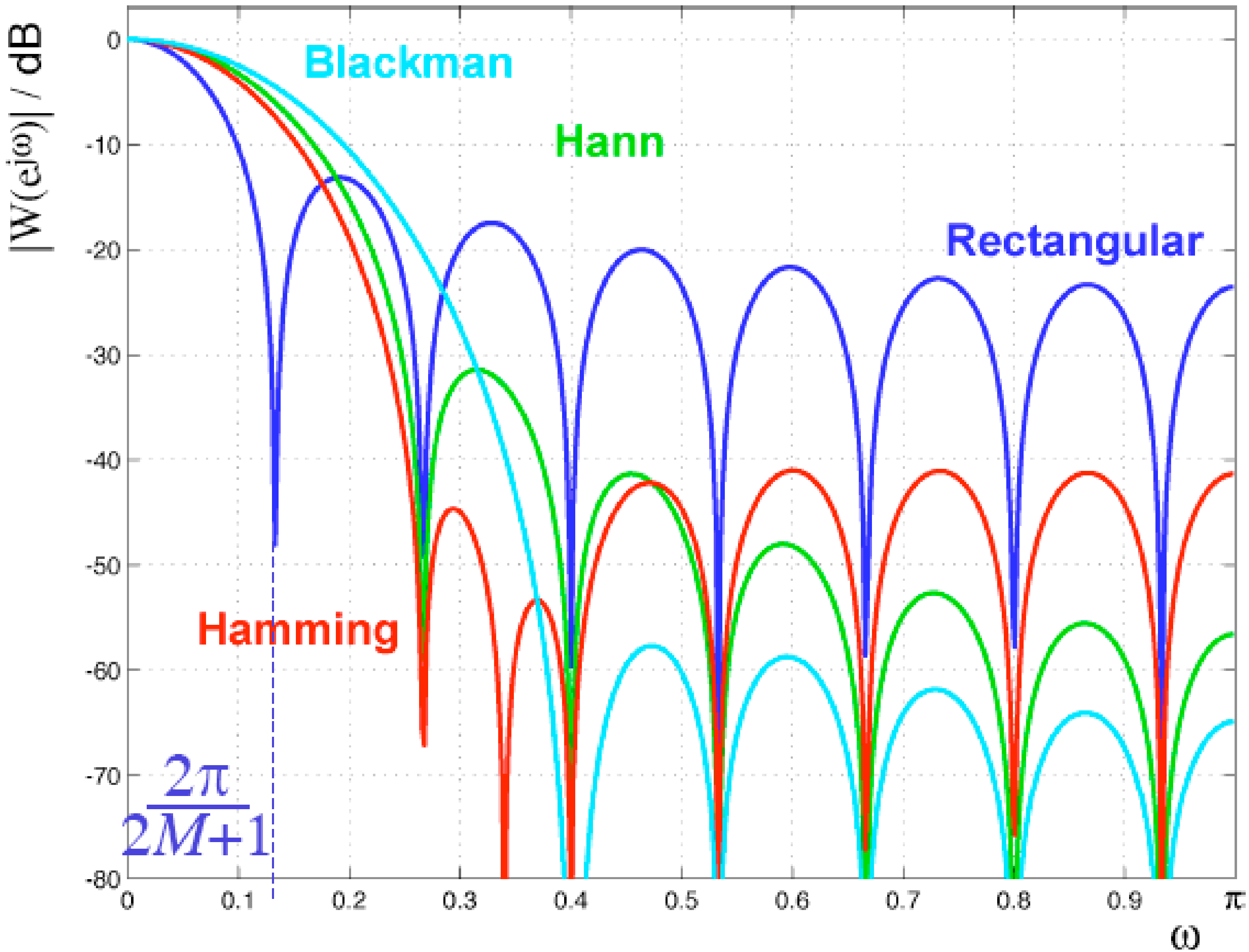
Rectangular

Hanning

Hamming

Blackman





|W(e^jω)| / dB
Blackman
Hann
Rectangular
Hamming
2π / 2M+1
0
-10
-20
-30
-40
-50
-60
-70
-80
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
π
ω

Windowed FIR Filter Example:

1. Get ideal filter impulse response:

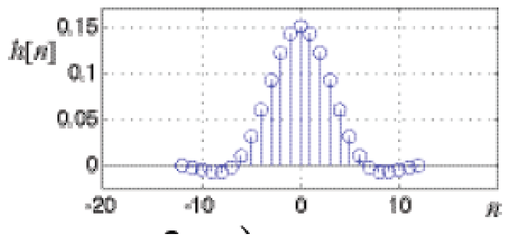
$$\omega_c = 0.15\pi \quad \Rightarrow h_d[n] = \frac{\sin 0.15\pi n}{\pi n}$$

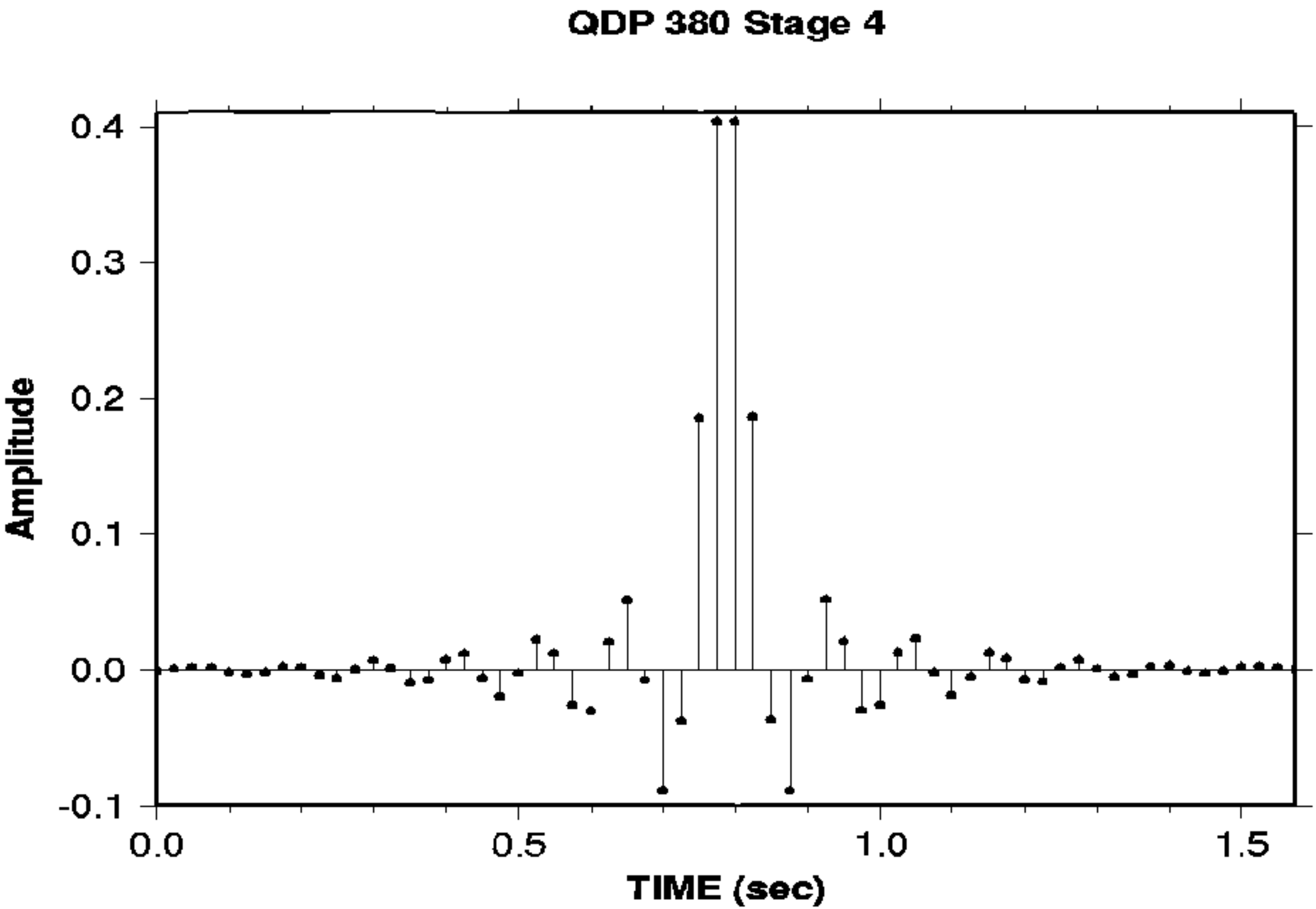
2. Get window function for truncation:

$$N = 25 \rightarrow M = 12 \quad \text{(N=2M+1)}$$
$$\Rightarrow w[n] = 0.54 + 0.46\cos\left(2\pi\frac{n}{25}\right) \quad -12 \le 12$$

3. Apply window:

$$h[n] = h_d[n]w[n]$$
$$= \frac{\sin 0.15\pi n}{\pi n}\left(0.54 + 0.46\cos\frac{2\pi n}{25}\right) \quad -12 \le 12$$




QDP 380 Stage 4
Amplitude
0.4
0.3
0.2
0.1
0.0
-0.1
0.0
0.5
1.0
1.5
TIME (sec)

Zero Phase FIR Filter

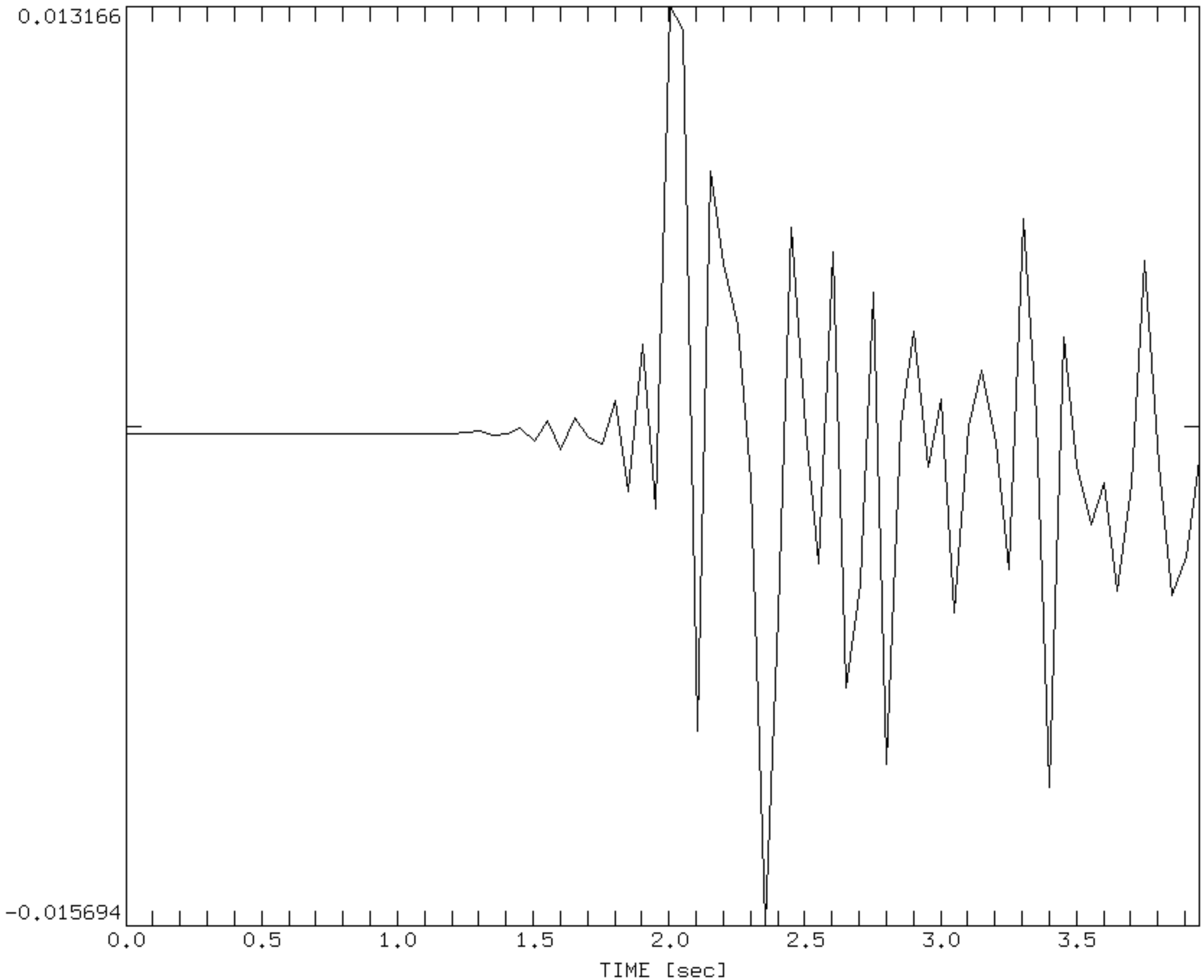
Problem: Two-Sided IR

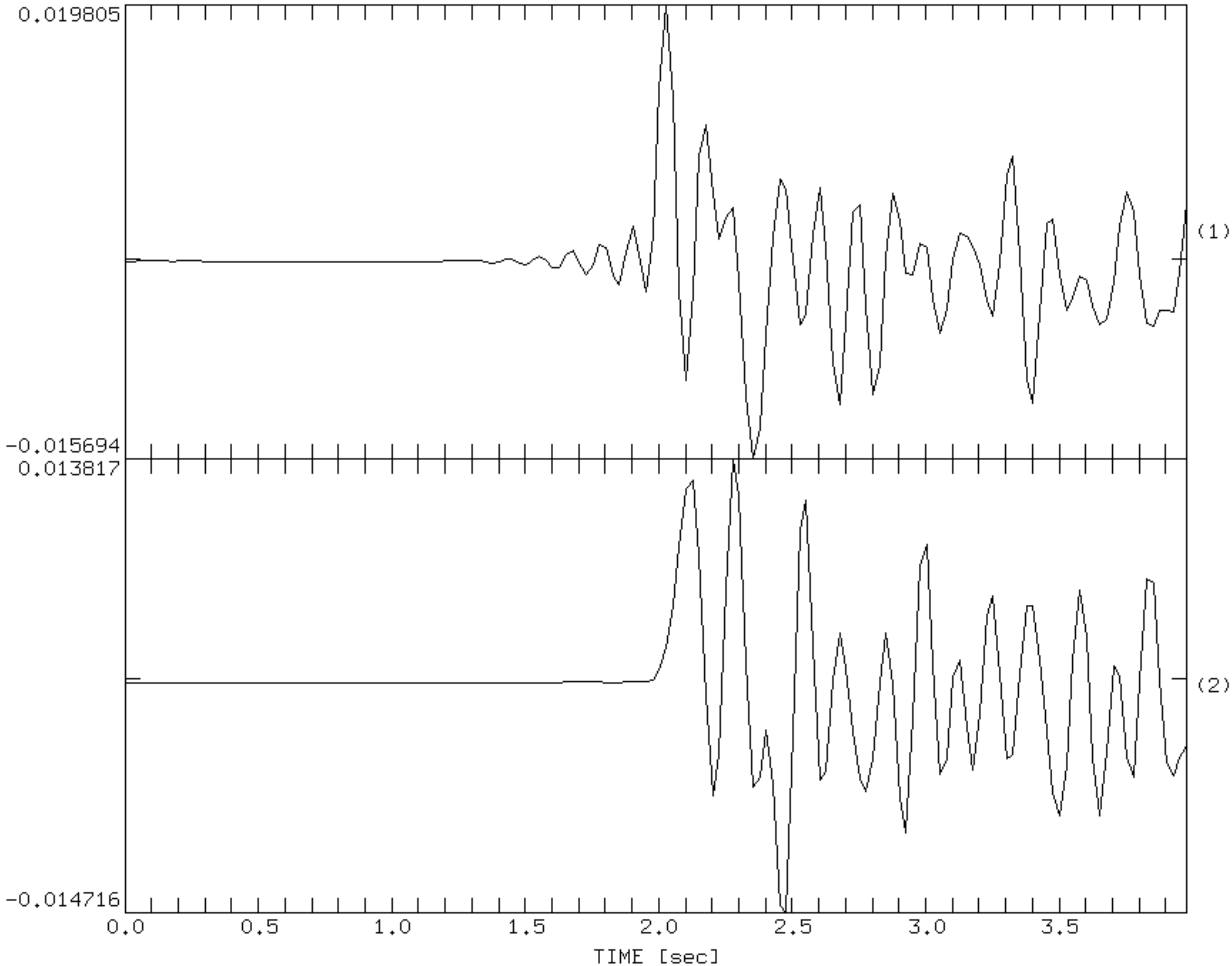
Cure: Change IR into Minimum Phase

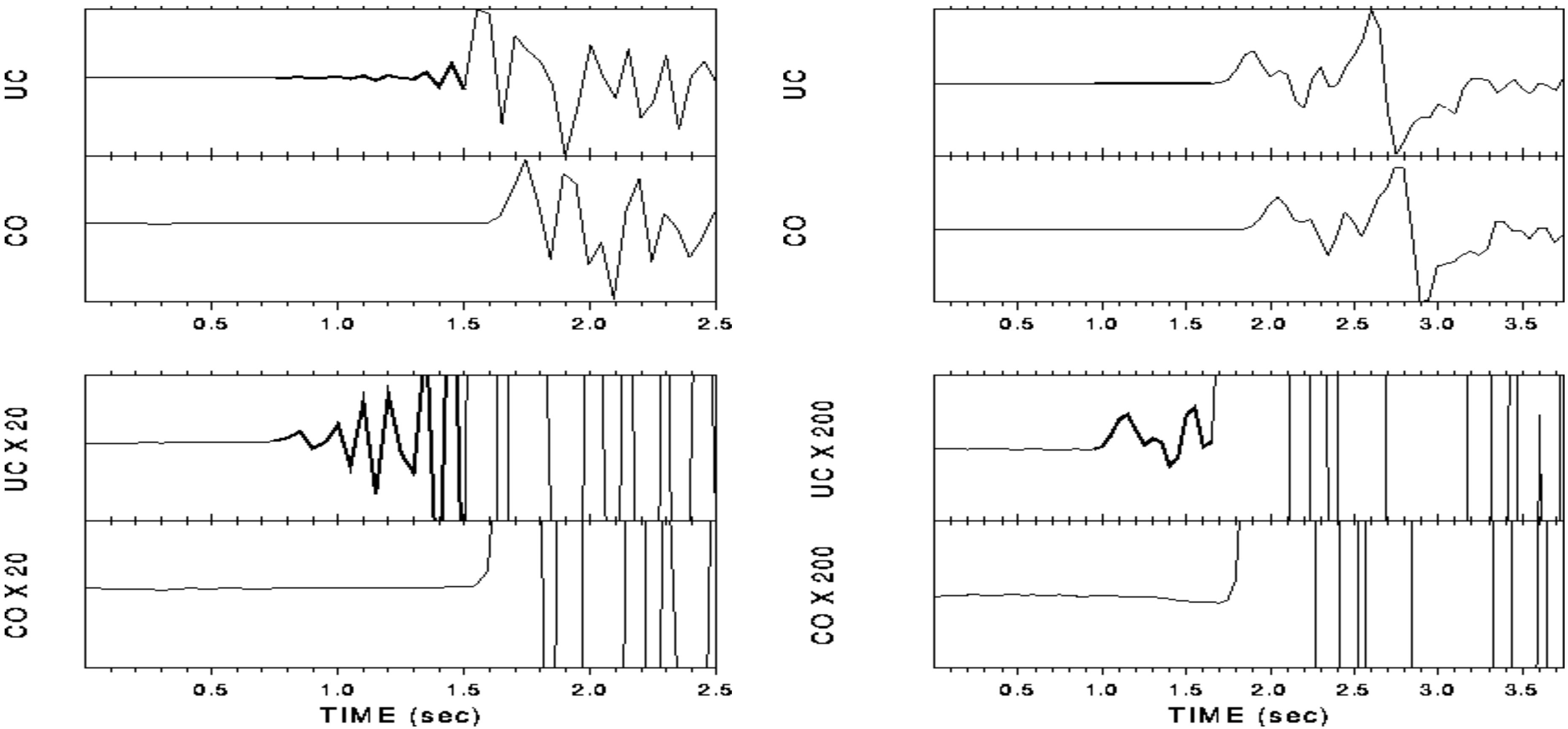
Methods:

1) Add phase of Minimum Phase Filter to trace spectrum

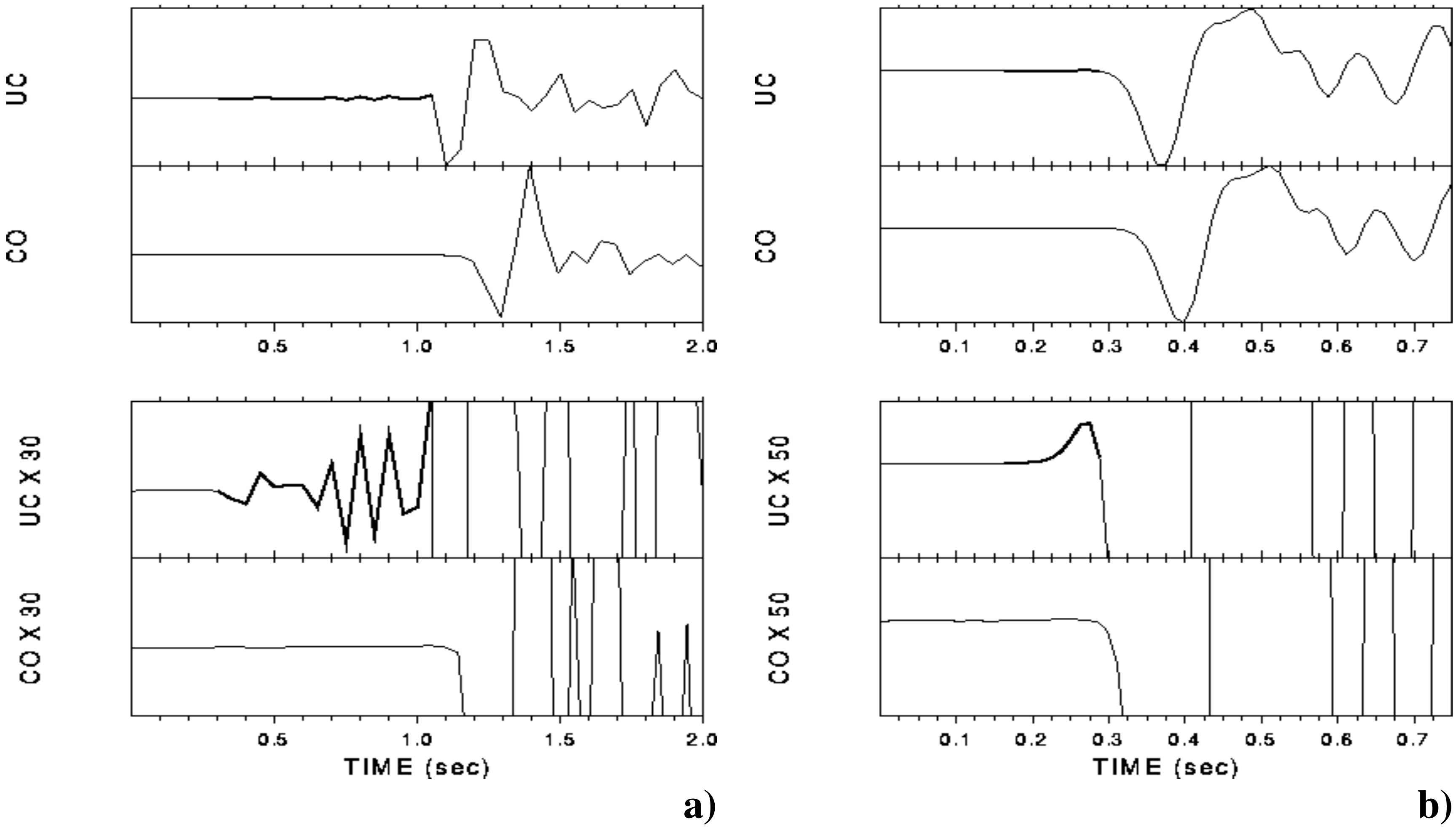
2) Recursive Filtering of time inverted trace

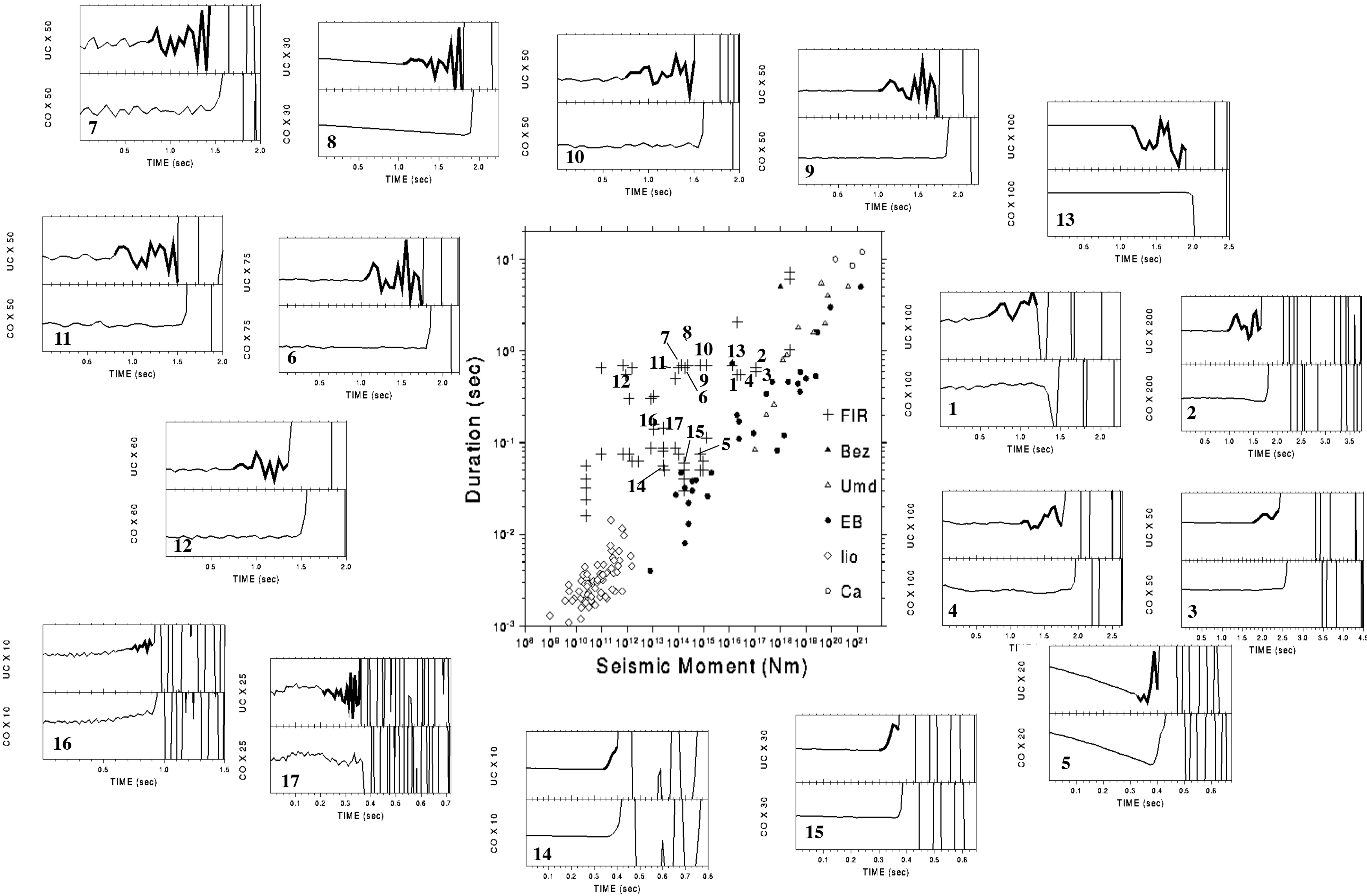

0.013166
-0.015694
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
TIME [sec]


0.019805
-0.015694
0.013817
-0.014716
(1)
(2)
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
TIME [sec]


UC
CO
0.5 1.0 1.5 2.0 2.5
UC X 20
CO X 20
0.5 1.0 1.5 2.0 2.5
TIME (sec)
UC
CO
0.5 1.0 1.5 2.0 2.5 3.0 3.5
UC X 200
CO X 200
0.5 1.0 1.5 2.0 2.5 3.0 3.5
TIME (sec)

a) **b)**


UC
CO
UC X 30
CO X 30
0.5
1.0
1.5
2.0
TIME (sec)
a)
UC
CO
UC X 50
CO X 50
0.1
0.2
0.3
0.4
0.5
0.6
0.7
TIME (sec)
b)


Duration (sec)
Seismic Moment (Nm)
FIR
Bez
Umd
EB
Iio
Ca
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16
17
UC X 50
CO X 50
UC X 30
CO X 30
UC X 100
CO X 100
UC X 200
CO X 200
UC X 75
CO X 75
UC X 60
CO X 60
UC X 10
CO X 10
UC X 25
CO X 25
UC X 20
CO X 20
TIME (sec)

Conclusions

FIR filter generated precursory artefacts:

- **can become impossible to be identified visually**
- **can have similar scaling properties as nucleation phases**

Zero - phase FIR filters in general

- **affect the determination of all onset properties (onset times, onset polar ities)**

Consequence

For the interpretation of onset properties (onset times, onset po larities, nucleation phases, etc.) the acausal response of the zero-phase FIR filter has to be removed

but not

for waveform analysis.