

General linear time invariant systems

Generalization of concepts

Rewrite the differential equation for the RC filter:

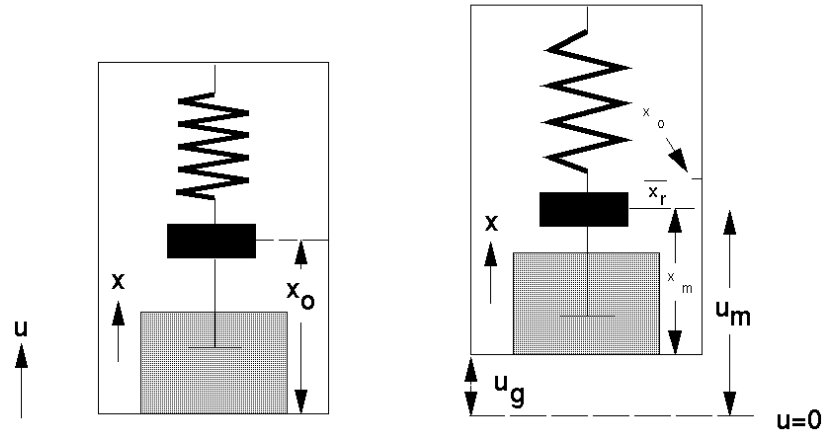
$$RC\dot{y}(t) + y(t) - x(t) = \alpha_1 \frac{d}{dt}y(t) + \alpha_0 y(t) + \beta_0 x(t) = 0$$

⇒ RC filter special case of
Nth order LTI system

$$\sum_{k=0}^L \alpha_k \frac{d^k}{dt} y(t) + \sum_{k=0}^L \beta_k \frac{d^k}{dt} x(t) = 0$$

WHAT IS IT GOOD FOR?????

Seismometer



- *The inertia of the mass* —

$$f_i = -m\ddot{u}_m(t)$$

- *The spring* —

$$f_{sp} = -kx_r(t)$$

k = spring constant (strength)

- *The dashpot* —

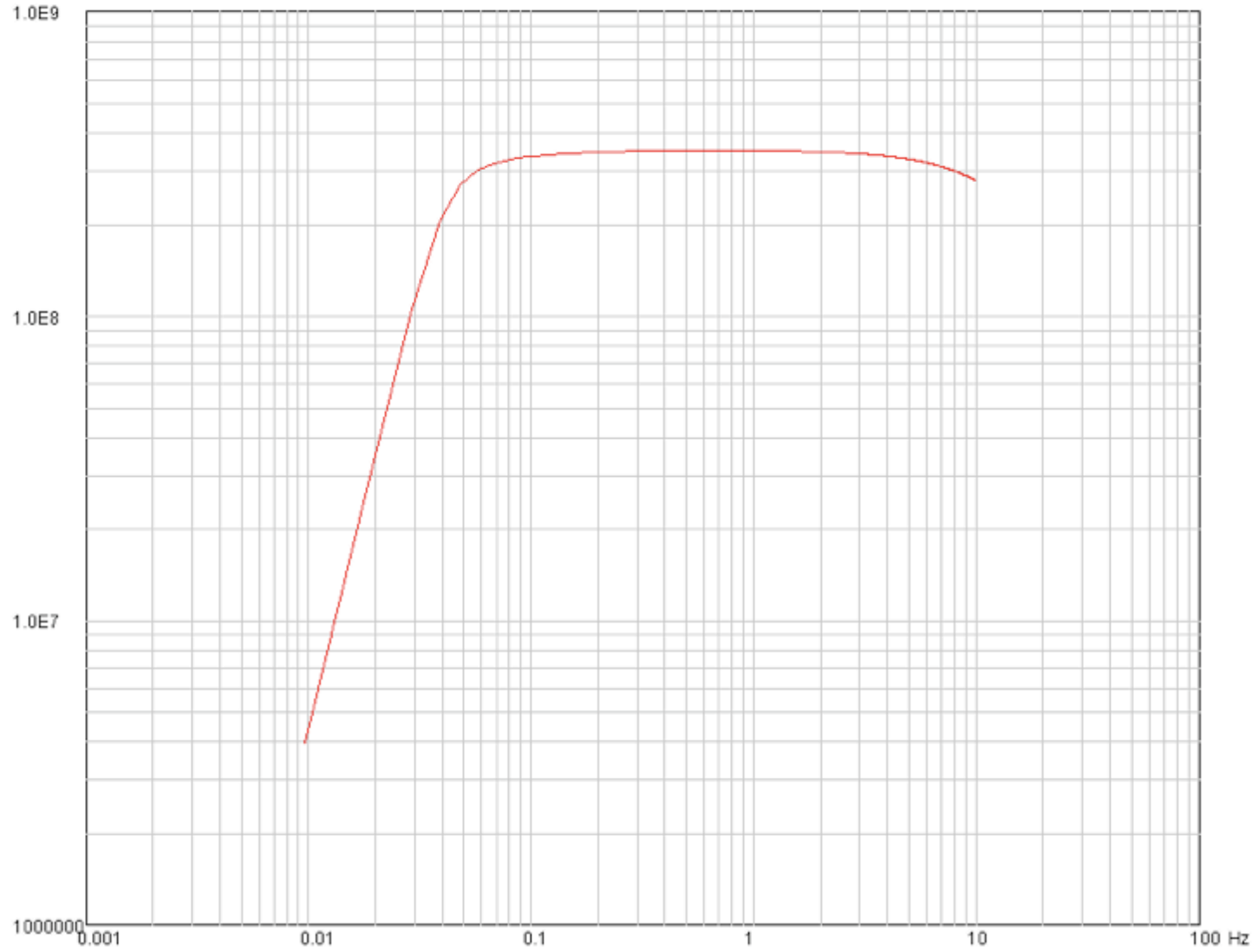
$$f_f = -D\dot{x}_m(t)$$

D = friction coefficient

$$-m\ddot{u}_m(t) - D\dot{x}_m(t) - kx_r(t) = 0 \quad \text{2nd order LTI !!}$$

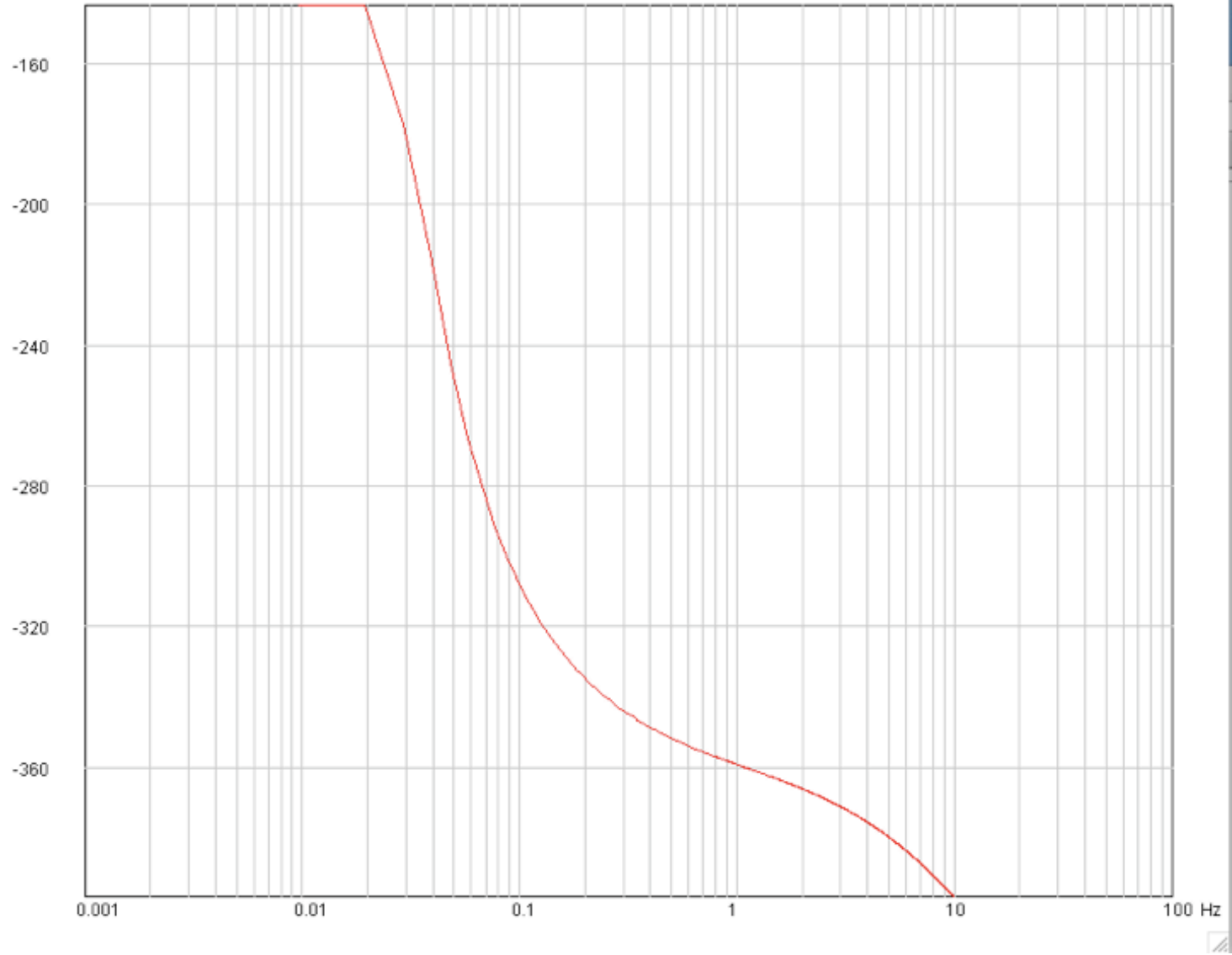


Amplitude





Phase



Poles and Zeros		
POLES	real part	imaginary part
	-0.1257	0.2177
	-0.1257	-0.2177
	-83.4473	0.0
	-0.3285	0.0
ZEROS	real part	imaginary part
	0.0	0.0
	0.0	0.0
	0.0	0.0
Scale Factor	2.90029998E10	

Problem 3.1

System with two poles. Consider three different cases:

- a) Put both poles at $-1.2566, 0$.
- b) Put one pole at location $-1.2566, 0$ and the other one at $1.2566, 0$.
- c) Put both poles at $1.2566, 0$.

For the input signal, use a spike at the center position of the window (for DST an internal sampling frequency of 100Hz and a window length of 2048 points works well). What types of impulse response functions do you expect in each case? Will the frequency response functions be different? What changes do you expect for the frequency response functions with respect to Problem 2.3 (single pole at $-1.2566, 0$)?

Consequences of transition from single pole to general N-th order system

- No major change in concept
- Zeros in addition to poles
- Can be treated in very similar way

Graphical estimation of the frequency response function

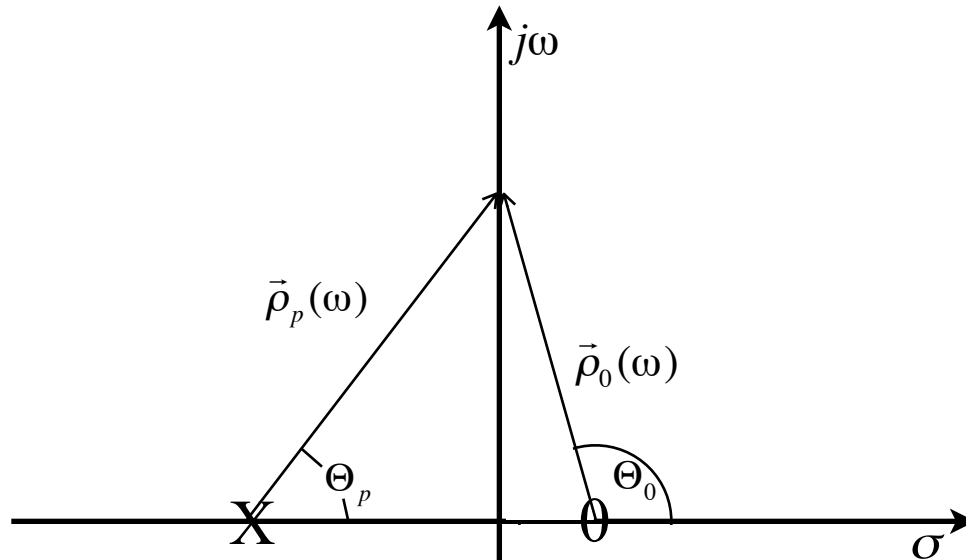


Fig. 3.2 Complex s plane representation of a system with a single pole and zero. The pole and zero locations are marked by an X, and a 0, respectively.

Transfer function: $T(s) = \frac{s - s_0}{s - s_p}$ s_0 and s_p : position of the zero and the pole, respectively.

Frequency response function: ($s = j\omega$) $T(j\omega) = \frac{j\omega - s_0}{j\omega - s_p}$

$j\omega - s_p \rightarrow$ vector $\vec{\rho}_p(\omega)$

$j\omega - s_0 \rightarrow$ vector $\vec{\rho}_0(\omega)$

$$T(j\omega) = |\rho_0(\omega)| e^{j\Theta_0} \frac{1}{|\rho_p(\omega)|} e^{-j\Theta_p}$$

Arbitrary LTI system

The **amplitude part** of the frequency response function of an arbitrary LTI system can be determined graphically by **multiplying the lengths of the vectors from the zero locations** in the S plane to the point $j\omega$ on the imaginary axis **divided by the product of all lengths of vectors from pole locations** to the point $j\omega$ on the imaginary axis. Likewise, to determine the **phase part**, the **phase angles for the vectors from the zero locations** in the S plane to the point $j\omega$ on the imaginary axis have to be **added** together. Then, the **phase angles of all the vectors from pole locations** to the point $j\omega$ on the imaginary axis have to be **subtracted**.

Problem 3.2

Use the argument given above to determine the frequency response for a system with a single pole at $-1.2566, 0$ if you add a zero at position $1.2566, 0$?

The phase properties of general LTI system

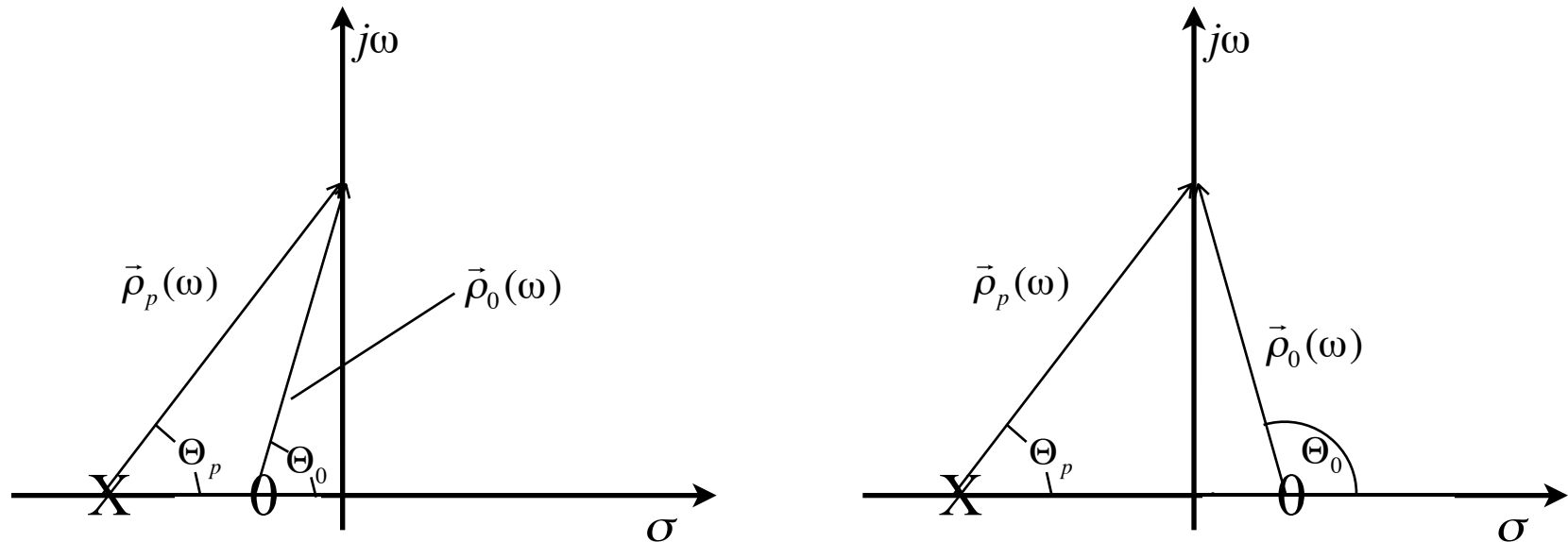


Fig. 3.3 Complex s plane representation of two systems with a single pole and zero. In b) the zero is at the same distance from the origin as in a) The pole and zero locations are marked by an X, and a 0, respectively.

How does the amplitude response differ?

How do the phase properties differ?

Minimum/maximum phase

A **causal stable** system (no poles in the right half plane) is minimum phase provided it has no zeroes in the right hand plane.

It is maximum phase if it has all its zeroes in the right hand plane.

minimum phase systems: nice properties !

Problem 3.3

How can the following two statements be proven for a general LTI system? a) If a system is minimum phase it will always have a stable and causal inverse filter. b) Any mixed phase system can be seen as a convolution of a minimum phase system and an allpass filter, which only changes the phase response but leaves the amplitude response as is.

Problem 3.4

How can we change the two-sided impulse response from Problem 3.1b (one pole at $-1.2566, 0$ and another one at $1.2566, 0$) into a right-sided one without changing the amplitude response? Keyword: allpass filter.

The interpretation of the frequency response function

A single pole in the transfer function causes the slope of the amplitude frequency response function to decrease by 20 dB/decade (6 dB/octave).

How about a zero?

A **single zero** causes an increase of the slope by the same amount .

The transition in either case occurs at corner frequencies which are equal to the modulus of the pole/zero position.

General rule

Each pole in the transfer function causes the slope of the amplitude frequency response function to decrease by 20 dB/decade (6 dB/octave).

Each zero causes an increase of the slope by the same amount .

The transition in either case occurs at corner frequencies which are equal to the moduli of the pole/zero positions.

Problem 3.5

Consider a system with a pole and a zero on the real axis of the s plane. Let the pole position be $(-6.28318, 0)$, and the zero position $(.628318, 0)$. What is the contribution of the zero to the frequency response function?

An internal sampling frequency of 20 Hz is recommended in DST

One more point...

- So far only singularities on the real axis. What happens for singularities away from real axis?

Are we allowed to do this?

Yes, but: Real systems require conjugate complex singularities.

Problem 3.6

Move the pole position of a double pole at $(-1.2566, 0)$ in steps of 15° (up to 75° and 85°) along the circle with the radius corresponding to the corner frequency (ω_p) of the poles. Calculate impulse response functions and the amplitude portions of the frequency response functions.

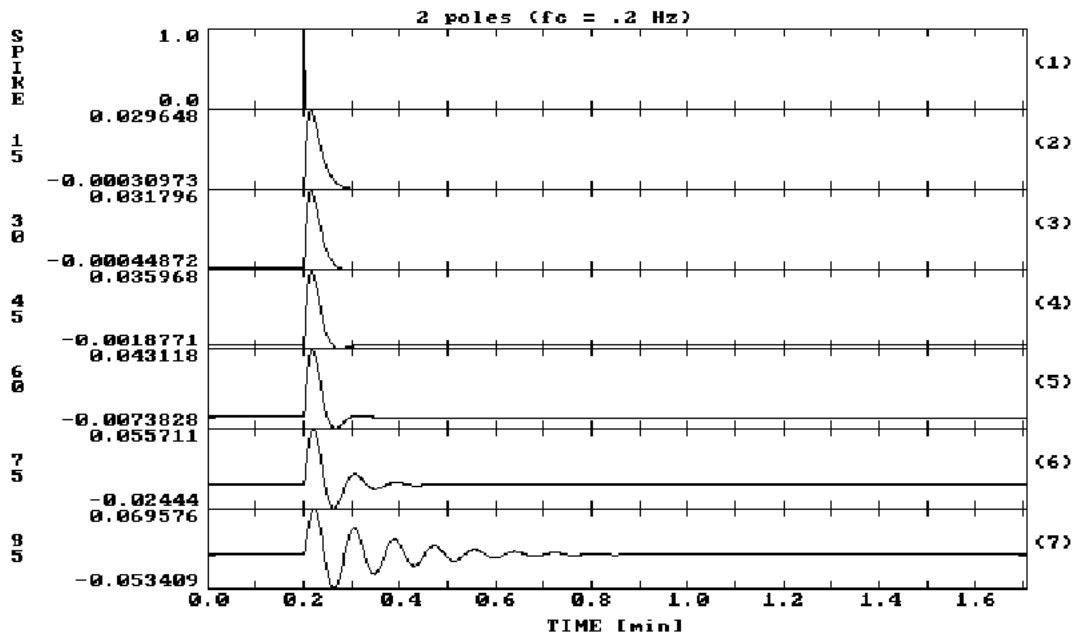


Fig. 3.5 Impulse responses for a system with a conjugate complex pole for the pole positions at different angles with the real axis of the s plane (15, 30, 45, 60, 75, 85°).

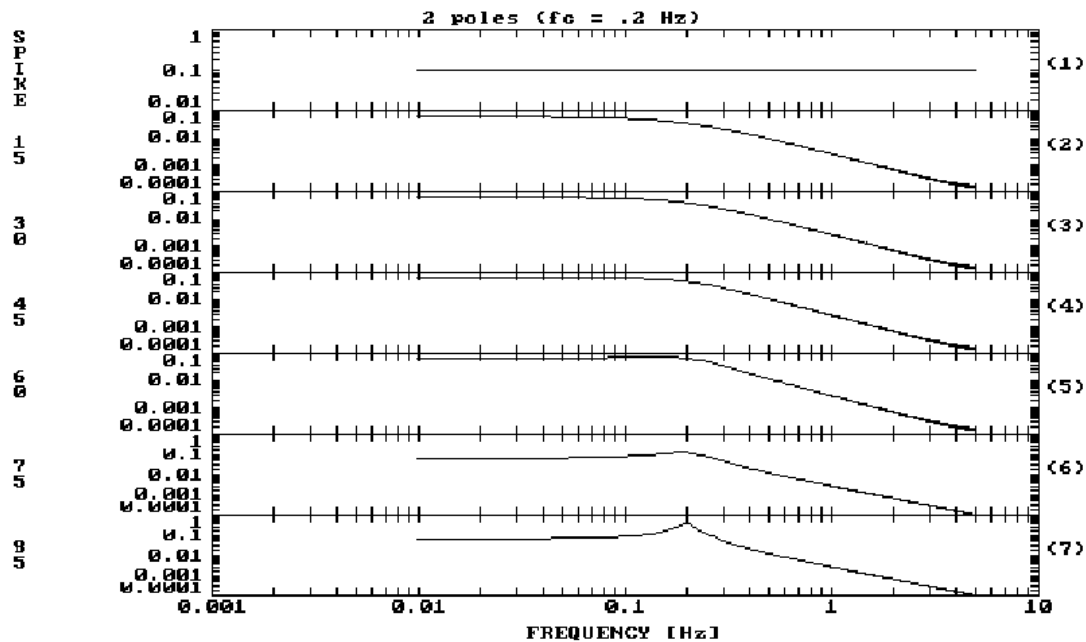
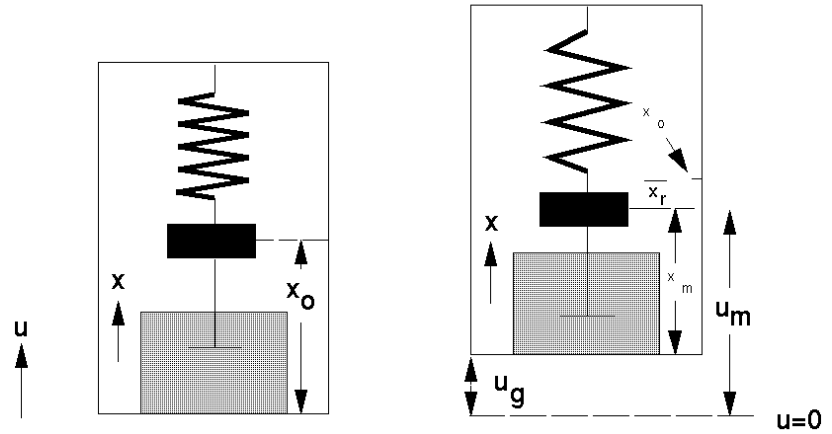


Fig. 3.6 Frequency response functions (amplitude) corresponding to Fig. 3.5.

Problem 3.7

Use the pole-zero approach to design a notch filter suppressing unwanted frequencies at 6.25 Hz. What kind of singularities do we need? How can we make use of the result of Problem 3.6?

Seismometer



- *The inertia of the mass* —

$$f_i = -m\ddot{u}_m(t)$$

- *The spring* —

$$f_{sp} = -kx_r(t)$$

k = spring constant (strength)

- *The dashpot* —

$$f_f = -D\dot{x}_m(t)$$

D = friction coefficient

Seismometer Transfer function

Laplace transform of $\ddot{x}_r(t) + 2\varepsilon\dot{x}_r(t) + \omega_0^2x_r(t) = -\ddot{u}_g(t)$

$$\Rightarrow s^2X_r(s) + 2\varepsilon sX_r(s) + \omega_0^2X_r(s) = -s^2U_g(s)$$

and $(s^2 + 2\varepsilon s + \omega_0^2)X_r(s) = -s^2U_g(s)$

Transfer function: $T(s) = \frac{X_r(s)}{U_g(s)} = \frac{-s^2}{s^2 + 2\varepsilon s + \omega_0^2}$

Since a quadratic equation $x^2 + bx + c = 0$ has the roots

$$x_{1,2} = -\frac{b}{2} \pm \sqrt{\frac{b^2}{4} - c}$$

pole positions $p_{1,2}$:

$$\begin{aligned}
 p_{1,2} &= -\varepsilon \pm \sqrt{\varepsilon^2 - \omega_0^2} \\
 &= -h\omega_0 \pm \omega_0 \sqrt{h^2 - 1} \\
 &= -\left(h \pm \sqrt{h^2 - 1}\right)\omega_0
 \end{aligned}$$

For the underdamped case ($h < 1$) the pole position becomes

$$p_{1,2} = -\left(h \pm j\sqrt{1 - h^2}\right)\omega_0$$

with the pole distance from the origin

$$|p_{1,2}| = \left| \left(h \pm j\sqrt{1 - h^2} \right) \right| \cdot |\omega_0| = \sqrt{h^2 + (1 - h^2)} \cdot |\omega_0| = |\omega_0|$$

Therefore, the poles of an underdamped seismometer are located in the left half of the s plane in a distance of $|\omega_0|$ from the the origin. The quantity $h |\omega_0|$ gives the distance from the imaginary axis.

Inverse and simulation filtering of digital seismograms

NEXT: From filter problem to the simulation problem - the conversion of digital (broad-band) records into those from different seismograph systems.

REASON: Signal amplitudes or onset time determination in a manner consistent with other observatories. Simulated systems will most commonly belong to the standard classes of instruments described by Willmore (1979) because **there is no single, optimum class of instruments for the detection and analysis of different types of seismic waves.**

Instrument classes

- High frequency teleseismic body waves: SP-instruments (class A)
- LP body waves and teleseismic surface waves: LP-instruments (class B)
- Regional body and surface waves: intermediate band (class C).
- Local magnitude: Wood-Anderson instrument.

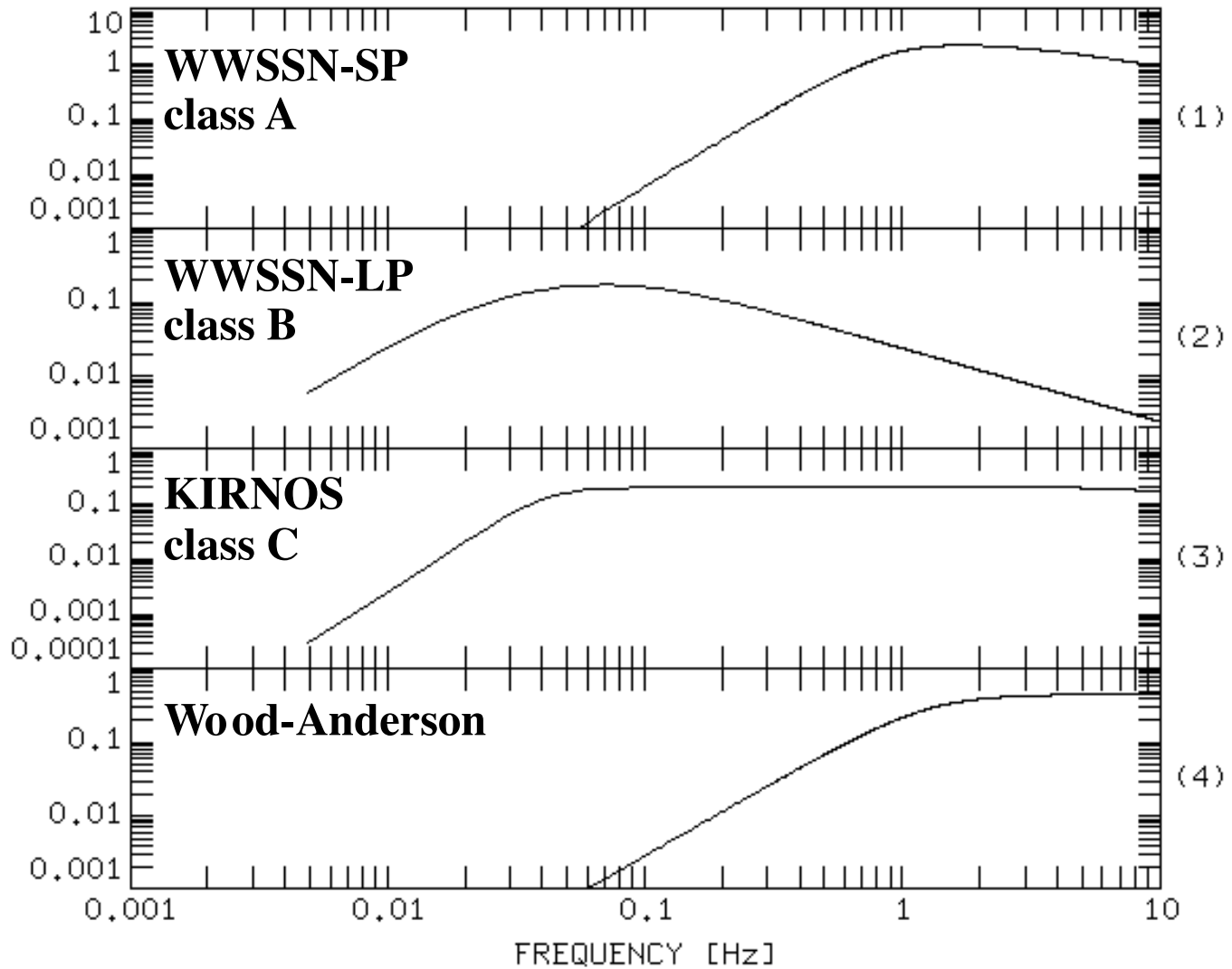


Fig. 9.1 Shapes of the moduli of the displacement frequency response functions of commonly used standard seismograph systems.

Simulation = Deconvolution + Filtering

$$Y_{sim}(z) = \frac{T_{syn}(z)}{T_{act}(z)} \cdot Y_{act}(z) = T_{sim}(z) \cdot Y_{act}(z)$$

$T_{act}(z)$ = transfer function of actual recording system

$T_{syn}(z)$ = transfer function of the instrument to be synthesized

$Y_{act}(z)$ = z- transform of the recorded seismogram

$Y_{sim}(z)$ = z- transform of the simulated seismogram

The concept of instrument simulation

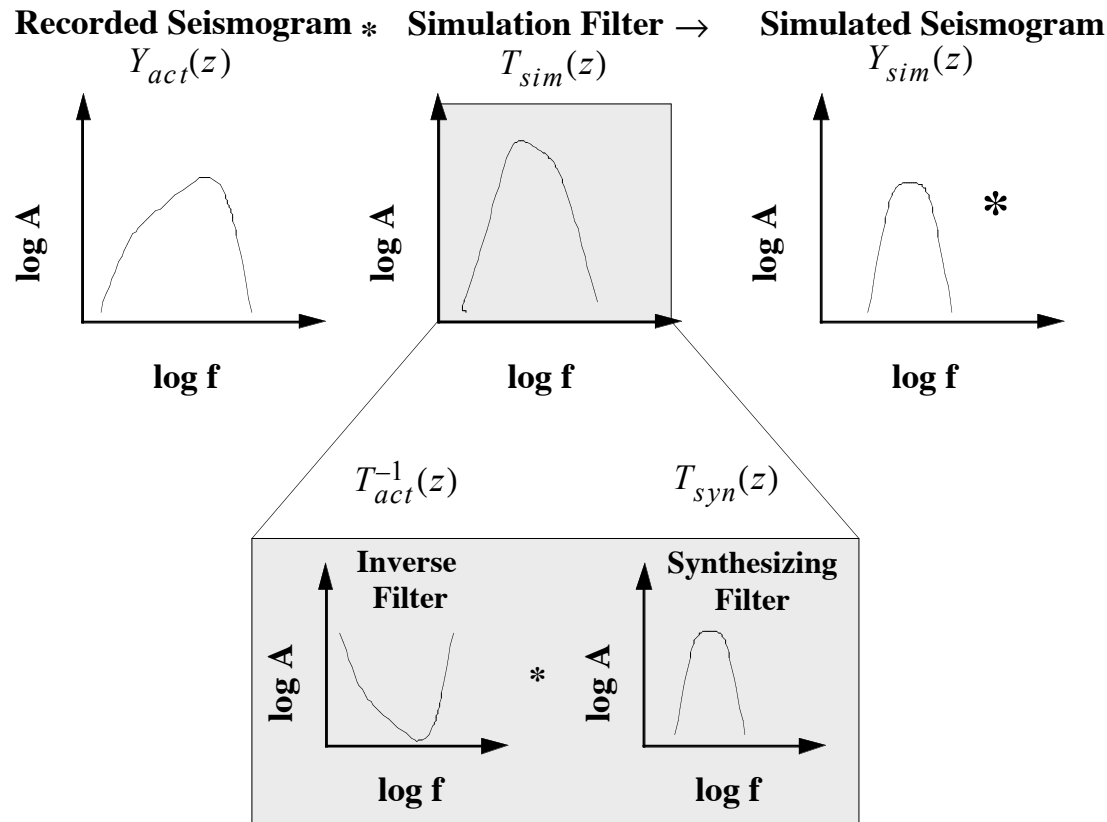


Fig. 9.2 The simulation of digital seismographs. The simulation filter can be thought of as a combination of an inverse filter for the actual recording system and a synthesizing filter for the simulated recording system. Displayed are schematic sketches of the amplitude frequency response functions of the contributing subsystems.

9.1 Stability problems

Noisefree situation

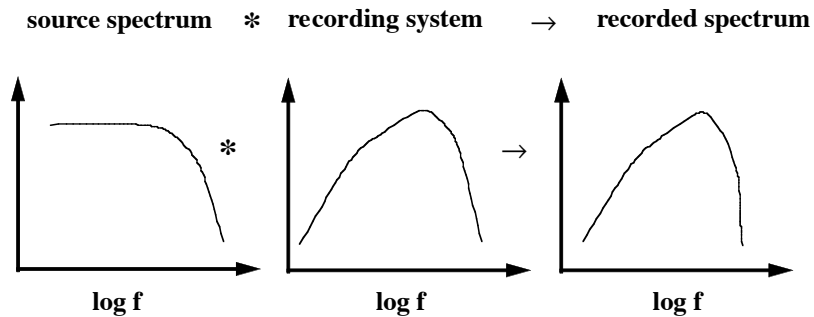


Fig. 9.3 Recording the displacement spectrum of an idealized earthquake source.

Recovery of source spectrum:

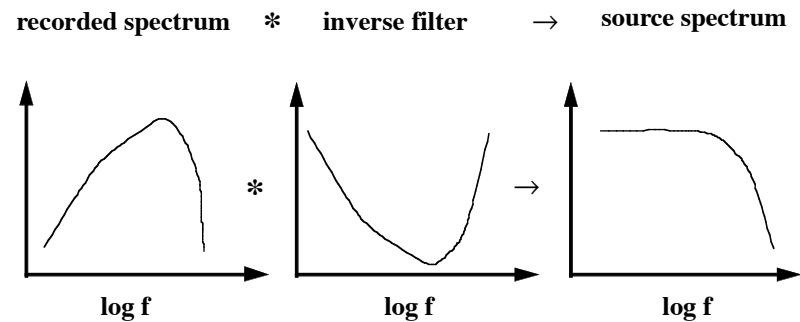


Fig. 9.4 Recovering the source spectrum by inverse filtering in the noise-free case.

Noisy situation

noisy' spectrum * inverse filter → 'noisy' source spectrum

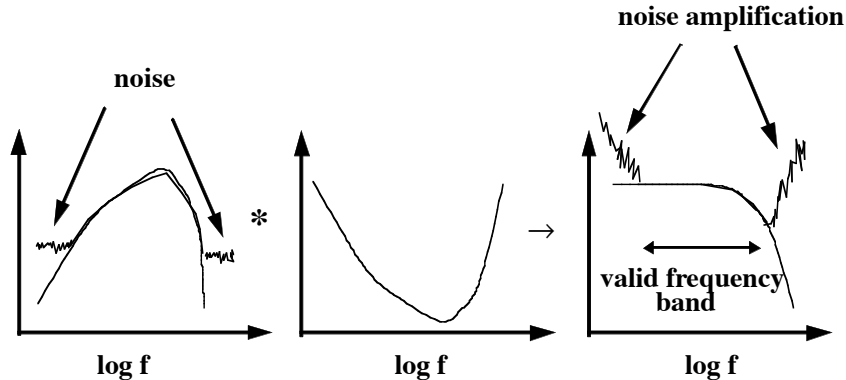


Fig. 9.5 Noise amplification by inverse filtering. The solid line in the left panel shows the signal plus noise while the noise-free signal is shown by the dashed line.

Problem:

- Decrease of signal-to-noise ratio (SNR) outside the pass-band of the recording instrument
- magnification of the inverse filter is largest where SNR is smallest. Thus, noise in this frequency band will be amplified (instability!).

Consequence:

- The instrument response can only be deconvolved within a certain *valid frequency band* in the presence of noise. The valid frequency band depends on both the signal-to noise ratio and the slope of the frequency response function of the recording systems.

Problem 3.8

From the shape of the frequency response function in the figure determine the poles and zeroes of the corresponding transfer function.

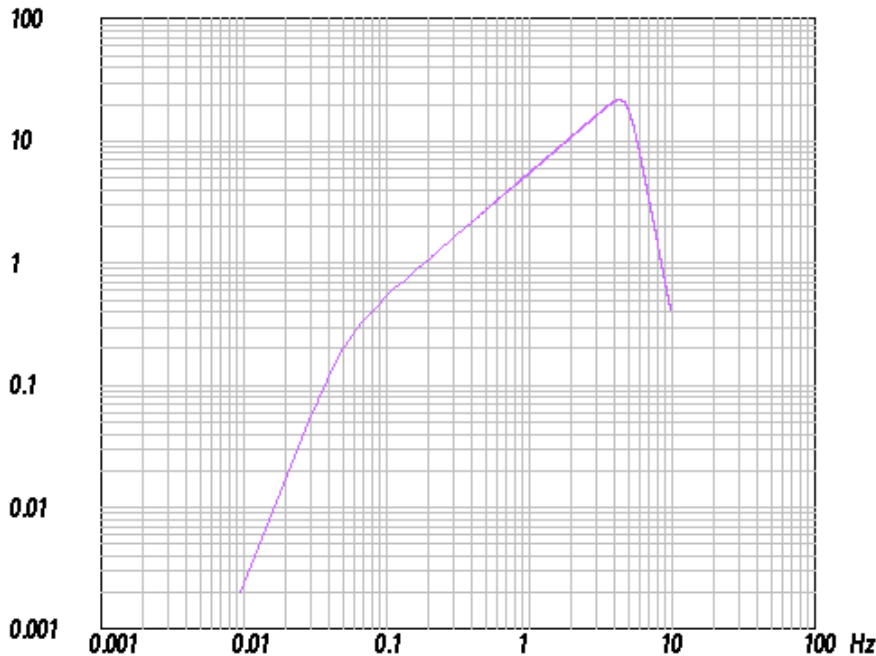


Fig. 3.4 Frequency response function (amplitude) with an unknown pole - zero distribution.

Solution 3.8 Within DST, the frequency response function of Problem 3.8 can be modeled by poles and zeros entered via the *Modify and Enter* option of the *Poles/ Zeros* menu. First select *problem3.8* from the *Load Response to Fit* option of the *Poles/Zeros* menu to display the frequency response function shown in Fig. 3.4. Next, estimate the different slopes and determine the number of poles and zeros which are needed to model them. One reasonable interpretation is sketched in Fig. A 3.15. Next, try to find the corner frequencies at which the changes in slope occur (here 0.05 Hz and 5 Hz).

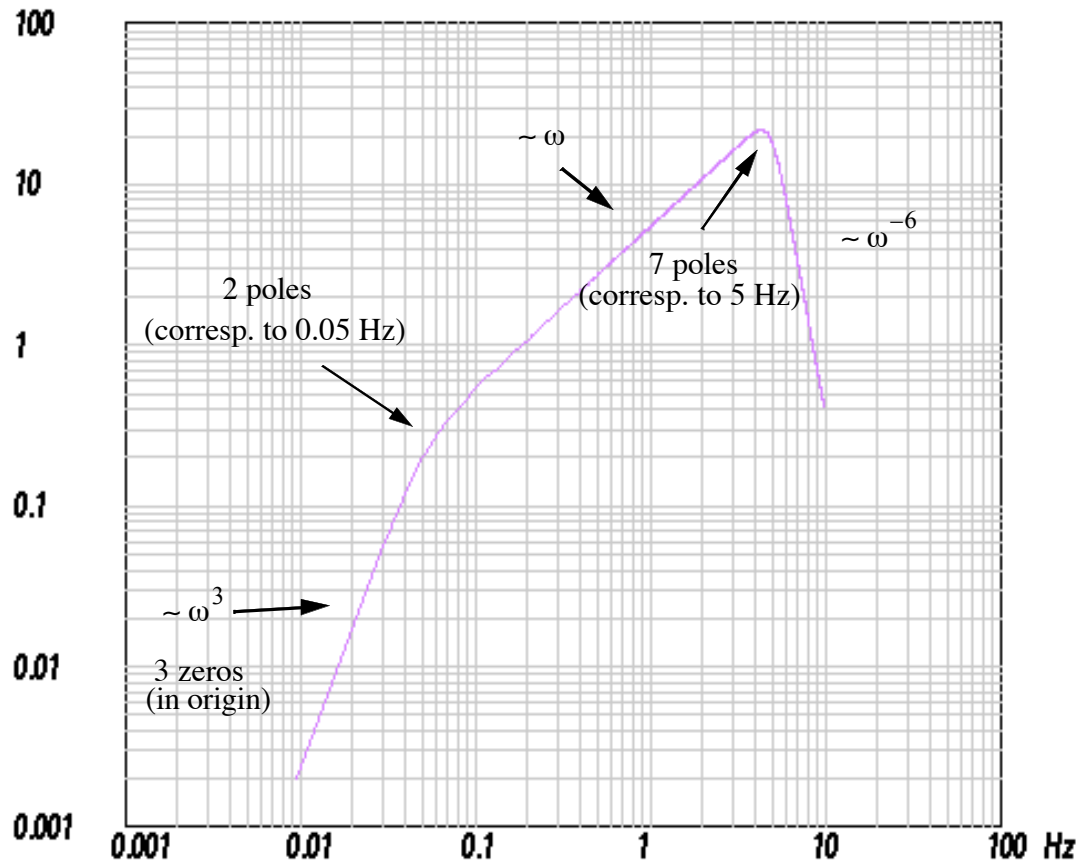


Fig. A 3.15 Frequency response function (amplitude) with an 'unknown' pole - zero distribution from Problem 3.8.

(-0.2221, 0.2221)
(-0.2221, -0.2221)
(-28.473, 13.277)
(-28.473, -13.277)
(-20.194, 24.066)
(-20.194, -24.066)
(-8.131, 30.346)
(-8.131, -30.346)
(-31.416, 0)

This is displayed below (Fig. A 3.16).

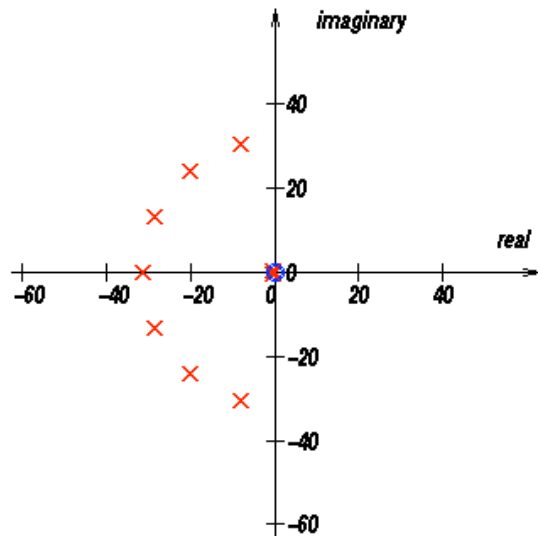


Fig. A 3.16 Trial pole/zero position to match the amplitude frequency response function of Problem 3.8.

At the corner frequency of 5 Hz, the frequency response function of Problem 3.8 has slightly higher amplitudes and shows a sharper change of slopes with respect to the trial frequency response function. This difference can be reduced by shifting some of the poles of the trial response closer to the imaginary axis to obtain a more "resonant" behaviour at that frequency.

The real distribution of poles and zeros for the frequency response function of Problem 3.8 is given below. It describes the frequency response function of the GRF array in SE Germany.

Poles:

(-0.2221, 0.2221)
(-0.2221, -0.2221)
(-7.0058, 30.6248)
(-7.0058, -30.6248)
(-19.5721, 24.5742)
(-19.5721, -24.5742)
(-28.3058, 13.6288)
(-28.3058, -13.6288)
(-31.4159, 0.0)

Zeros:

(0.0, 0.0)
(0.0, 0.0)
(0.0, 0.0)

Scale factor:

2.49059e10

For the generation of Fig. A 3.15 - Fig. A 3.17 within DST, an internal sampling frequency of 20 Hz and a window length of 2048 points was used.